FREE DAMPED VIBRATION

In many practical systems, the vibrational energy is gradually converted to heat or sound. Due to the reduction in the energy, the response, such as the displacement of the system, gradually decreases. The mechanism by which the vibrational energy is gradually converted into heat or sound is known as damping. Although the amount of energy converted into heat or sound is relatively small, the consideration of damping becomes important for an accurate prediction of the vibration response of a system. A damper is assumed to have neither mass nor elasticity, and damping force exists only if there is relative velocity between the two ends of the damper. It is difficult to determine the causes of damping in practical systems. Hence damping is modeled as one or more of the following types.

Types of Damping

1. Viscous damping
2. Coulomb damping
3. Structural damping
4. Slip or interfacial damping

1. Viscous damping

Viscous damping is the most commonly used damping mechanism in vibration analysis. When mechanical systems vibrate in a fluid medium such as air, gas, water, or oil, the resistance offered by the fluid to the moving body causes energy to be dissipated. In this case, the amount of dissipated energy depends on many factors, such as the size and shape of the vibrating body, the viscosity of the fluid, the frequency of vibration, and the velocity of the vibrating body. In viscous damping, the damping force is proportional to the velocity of the vibrating body. Typical examples of viscous damping include (1) fluid film between sliding surfaces, (2) fluid flow around a piston in a cylinder, (3) fluid flow through an orifice, and (4) fluid film around a journal in a bearing.

2. Coulomb damping

Here the damping force is constant in magnitude but opposite in direction to that of the motion of the vibrating body. It is caused by friction between rubbing surfaces that either are dry or have insufficient lubrication.

3. Structural damping

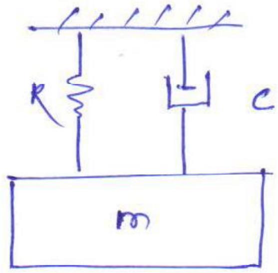
When a material is deformed, energy is absorbed and dissipated by the material. The effect is due to friction between the internal planes, which slip or slide as the deformations take place.

When a body having material damping is subjected to vibration, the stress-strain diagram shows a hysteresis loop. The area of this loop denotes the energy lost per unit volume of the body per cycle due to damping

4. <u>Slip or interfacial damping</u>

Microscopic slip occur on the interfaces of machine elements in contact under fluctuating loads. The amount of damping depends upon the material combination, surface roughness at interface, contact pressure and the amplitude of vibration.

Differential equation of free damped Vibration



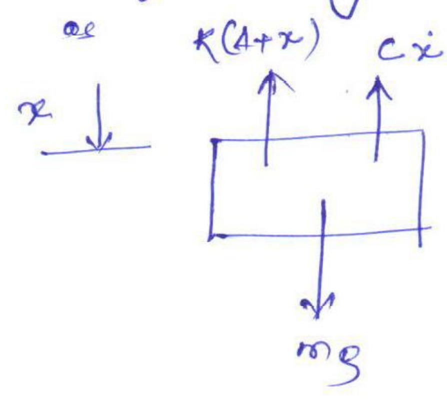
In the study of vibration, the process of energy dissipation is generally referred to as damping. The most common phenomenon of energy dissipating element is viscous damper, also called dashpot.

Viscous damping force is proportional to the velocity $\dot{x}$ of the mass and acts in the direction opposite to the velocity of the mass m. It can be expressed as,

$$F = c\dot{x} \quad \text{---(1)}$$

Where c = damping coefficient of viscous damping.

The free body diagram of the system can be represented as



Applying Newton's second law

$$m\ddot{x} = -K(\Delta + x) + mg - c\dot{x}$$

$$\Rightarrow m\ddot{x} = -\cancel{K\Delta} - Kx + \cancel{mg} - c\dot{x}$$

$$= -Kx - c\dot{x}$$

$$\Rightarrow \boxed{m\ddot{x} + c\dot{x} + Kx = 0} \quad \text{---(2)}$$

or $\quad \ddot{x} + \left(\frac{c}{m}\right)\dot{x} + \left(\frac{K}{m}\right)x = 0 \quad \text{---(3)}$

Eq.(3) is the differential equation of motion for free vibration of a damped spring-mass system.

Assuming a solution in the form $x(t) = Ce^{st}$ to obtain the auxiliary equation

$$s^2 + \frac{c}{m}s + \frac{K}{m} = 0 \quad \text{---(4)}$$

Eq.(4) has roots

$$s_{1,2} = \frac{1}{2}\left[-\frac{c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 - 4\frac{K}{m}}\right]$$

or $$\boxed{s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{K}{m}}} \quad (5)$$

The solution of eq.(5) takes one of the three forms, depending on whether the quantity $\left[\left(\frac{c}{2m}\right)^2 - \frac{K}{m}\right]$ is zero, positive or negative.

If $\left(\frac{c}{2m}\right)^2 - \frac{K}{m} = 0$ we have,

$$\frac{c}{2m} = \sqrt{\frac{K}{m}} = \omega_n$$

$$\Rightarrow \boxed{c = 2m\omega_n} \quad (6)$$

in which case we have repeated roots

$s_1 = s_2 = -\frac{c}{2m}$ and the solution is

$$\boxed{x(t) = (A+B)\,e^{-(c/2m)t}} \quad (7)$$

In this particular case, the damping constant or coefficient is called critical damping constant denoted by

$$\boxed{c_c = 2m\omega_n} \quad (8)$$

And eq.(5) may be written as

$$s_{1,2} = -\frac{c}{c_c}\omega_n \pm \omega_n\sqrt{\left(\frac{c}{c_c}\right)^2 - 1}$$

or $$\boxed{s_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n} \quad (9)$$

Where $\omega_n = \sqrt{\frac{k}{m}}$, circular frequency of the corresponding undamped system and

$$\boxed{\zeta = \frac{c}{c_c} = \frac{c}{2m\omega_n}} \quad \text{—— (10)}$$

and ζ = damping factor.

<u>Case - 1</u> when $\zeta < 1$

If $\zeta < 1$ both the roots in eq. (9) are imaginary and given by

$$s_{1,2} = \left(-\zeta \pm i\sqrt{1-\zeta^2}\right)\omega_n \quad \text{—— (11)}$$

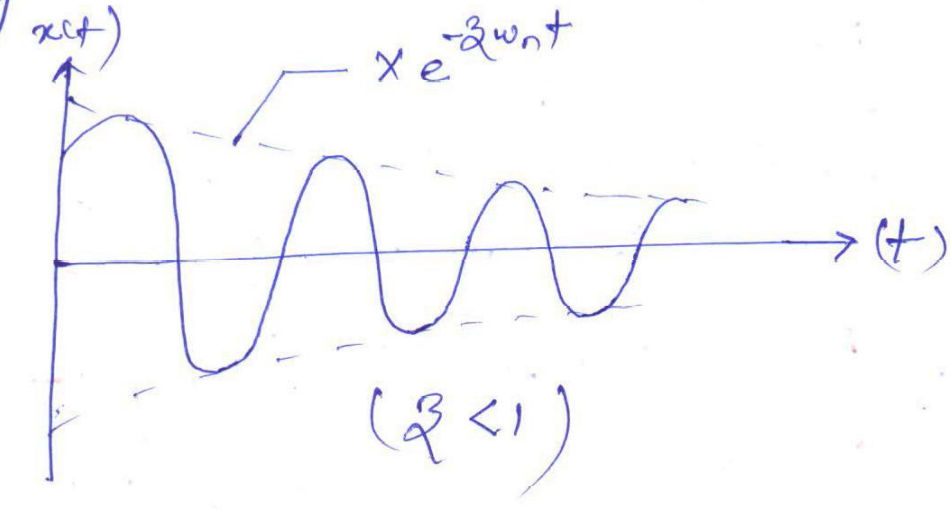
and the solution of motion is

$$\boxed{x(t) = X e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)} \quad \text{—— (12)}$$

where ω_d = damped circular frequency (which is always less than ω_n)

ϕ = phase angle of damped oscillation.

The function is a harmonic function whose amplitude decays exponentially with time. The general form of motion is shown in the figure and the system is said to be underdamped.



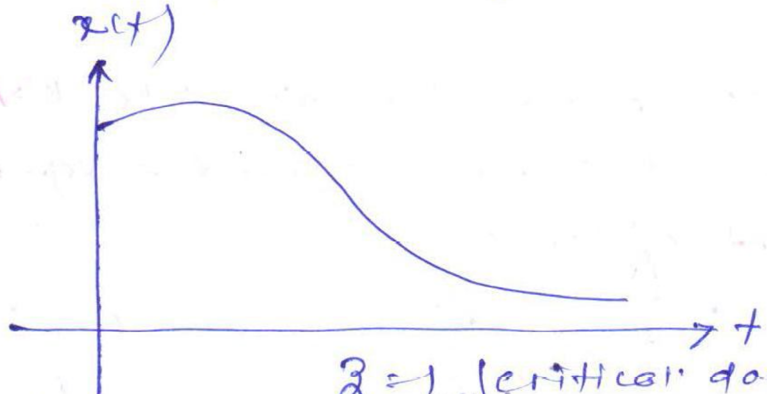
$(\zeta < 1)$

Case 2:- $\zeta = 1$ or $c = c_c = 2m\omega_n$

If $\zeta = 1$, the damping constant is equal to the critical damping constant and the system is called to be critically damped.

The displacement equation (7) may be written as

$$x(t) = (A + Bt)e^{-\omega_n t} \quad \text{--- (13)}$$



$\zeta = 1$ (critical damping)

The solution to the above equation (13) is the product of a linear function of time and decaying exponential.

Case - 3 $\zeta > 1$ or $c > 2m\omega_n$

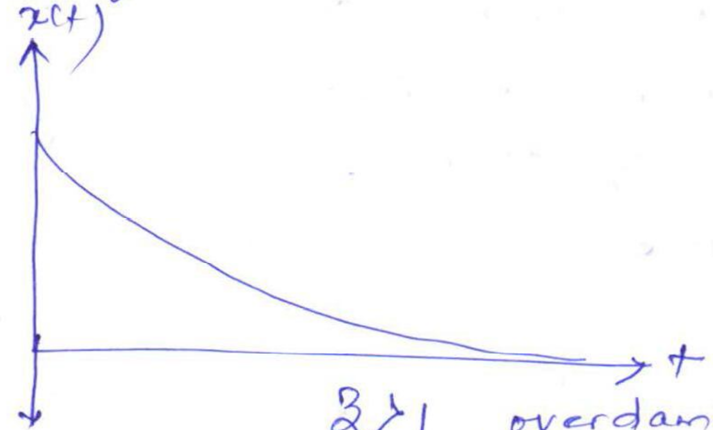
If $\zeta > 1$ the system is called overdamped. Here both the roots are real and are given by

$$s_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n$$

Since $\sqrt{\zeta^2 - 1} < \zeta$, it can be seen that both s_1 and s_2 are negative so that the displacement is the sum of two decaying exponentials given by

$$\boxed{x(t) = C_1 e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} + C_2 e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t}} \quad \text{--- (14)}$$

The motion will be non oscillating and shown in figure.



$\zeta > 1$, overdamping.

Example - 1

A damped spring-mass has $m = 12$ kg $k = 12$ N/mm and $c = 0.3$ N s/mm. Obtain the equation of displacement of the mass.

The natural frequency of the undamped system is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{12 \times 1000}{12}} = 31.62 \text{ rad/sec}$$

Critical damping constant $c_c = 2m\omega_n$

$$= 2 \times 12 \times 31.62 = 758.95 \text{ Ns/m or}$$

$$0.759 \text{ Ns/mm},$$

And damping factor $\zeta = \frac{c}{c_c} = \frac{0.3}{0.759} = \boxed{0.395}$

As the system is underdamped ($\because \zeta < 1$)

the damped natural frequency $\omega_d = (\sqrt{1-\zeta^2})\omega_n$

$$= \{\sqrt{1-(0.395)^2}\} 31.62 = 29.05 \text{ rad/s.}$$

and $\zeta\omega_n = 0.395 \times \cancel{30} 31.62 = 11.47$

Equation of displacement

$$x(t) = Xe^{-11.47t} \sin(29.05t + \phi)$$

$$\boxed{Xe^{-\zeta\omega_n t} \sin(\omega_d t + \phi)}$$

Example - 2

A single dof viscously damped system has a spring stiffness of 6000 N/m, critical damping constant of 0.3 Ns/mm and a damping ratio of 0.3. If the system is given an initial velocity of 1 m/s, determine the max^m displacement of the system.

The natural frequency of the system $\omega_n = \sqrt{\frac{k}{m}}$

We have $c_c = 0.3\ Ns/mm = 300\ Ns/m = 2m\omega_n$

$$= 2m\sqrt{\frac{k}{m}} = 2\sqrt{6000m}$$

$$\Rightarrow 300 = 2\sqrt{6000m} \Rightarrow m = \boxed{3.75\ kg.}$$

$$\omega_n = \sqrt{\frac{6000}{3.75}} = 40\ rad/sec.$$

Damping ratio $\zeta = \frac{\omega}{\omega_n} = 0.3$

or $c = c_c \times 0.3 = 0.3 \times 0.3 = 0.09\ Ns/mm$

$= 900\ Ns/m.$

Assuming $x_0 = 0$. and $\dot{x}_0 = 1\ m/s$. the general expression for displacement is

$$x(t) = e^{-\zeta\omega_n t}\frac{\dot{x}_0}{\omega_n\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\,\omega_n t$$

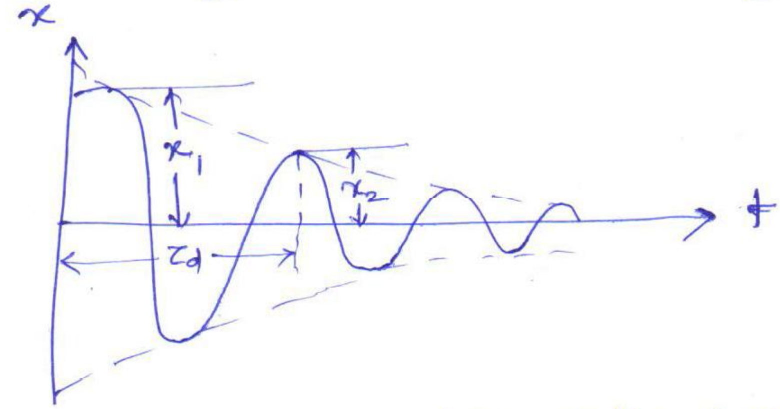
for max^m displacement (x_{max}) $\omega_n t = \pi/2$

and $\sin\sqrt{1-\zeta^2}\,\omega_n t = 1$

$$x_{max} = e^{-0.3\times\left(\frac{\pi}{2}\right)}\frac{1}{40\sqrt{1-0.3^2}}(1) = \boxed{0.01636\ m}$$

Logarithmic Decrement :-

The logarithmic decrement represents the rate at which the amplitude of a free damped vibration decreases, It is defined as the ratio of any two successive amplitudes on the same side of the mean line. ~~bottom of~~



In othe words we can say it's defined as the natural logarithm of the ratio of any two successive amplitudes. The displacement of an underdamped system is a sinusoidal oscillation with decaying amplitude as shown in the figure

The ratio of successive amplitude is

$$\boxed{\frac{x_i}{x_{i+1}} = \frac{Xe^{-\xi\omega_n t_i}}{Xe^{-\xi\omega_n(t_i+\tau_d)}} = e^{\xi\omega_n\tau_d} = \text{constant}} \quad \text{--- (1)}$$

So

$$\boxed{\frac{x_i}{x_{i+1}} = e^{\xi\omega_n\tau_d}} \quad \text{--- (2)}$$

Now substituting $\tau_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\xi^2}}$ in eq.(2)

$$\frac{x_i}{x_{i+1}} = e^{\xi \cdot \omega_n \cdot \frac{2\pi}{\omega_n\sqrt{1-\xi^2}}} = e^{\left(2\pi\xi/\sqrt{1-\xi^2}\right)}$$

and for small damping

$$\boxed{\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}} = 2\pi\xi} \quad \text{--- (3)}$$

$\because$ If ζ is small then $\delta = 2\pi\zeta$

since $\sqrt{1-\zeta^2} \cong 1$

From equation (3) we have

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

or $\zeta = \dfrac{\delta \cdot \sqrt{1-\zeta^2}}{2\pi}$

or $\zeta^2 = \dfrac{\delta^2(1-\zeta^2)}{(2\pi)^2}$

$\Rightarrow (2\pi)^2 \cdot \zeta^2 = \delta^2 - \delta^2\zeta^2$

$\Rightarrow (2\pi)^2 \cdot \zeta^2 + \delta^2\zeta^2 = \delta^2$

$\Rightarrow \zeta^2 \left[(2\pi)^2 + \delta^2\right] = \delta^2$

$$\Rightarrow \boxed{\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}} \quad \text{--- (4)}$$

Also $\boxed{\zeta = \dfrac{\delta}{2\pi}}$ (For small damping) — (5)

- Logarithmic decrement can also be calculated from the ratio of amplitudes of several cycles apart.

Thus if x_n is the amplitude n cycles after x_0, then

$$\frac{x_0}{x_n} = \frac{x_0}{x_1} \cdot \frac{x_1}{x_2} \cdot \frac{x_2}{x_3} \cdots \frac{x_{n-1}}{x_n}$$

$$= \left(\frac{x_j}{x_{j+1}}\right)^n$$

Natural log of the ratio $\ln\left(\dfrac{x_0}{x_n}\right) = n \ln\left(\dfrac{x_j}{x_{j+1}}\right)$

$$\Rightarrow \boxed{\ln\left(\frac{x_0}{x_n}\right) = n \cdot \delta} \quad \text{--- (6)}$$

or $\boxed{\delta = \frac{1}{n} \ln\left(\frac{x_0}{x_n}\right)}$ —(7)

so logarithmic decrement δ can be obtained from the amplitude loss occuring over several cycles

$$\boxed{n = \frac{1}{\delta} \ln\left(\frac{x_0}{x_n}\right) = \frac{\sqrt{1-\zeta^2}}{2\pi\zeta} \ln\left(\frac{x_0}{x_n}\right)} \quad -(8)$$

Equation (8) is used to determine no of cycles required for a given system to reach a specified reduction in amplitude

Example A single dof viscous damping system makes 5 complete oscillations/second. Its amplitude diminishes to 15% in 60 cycles. Determine

(a) logarithmic decrement

(b) damping ratio.

(a) Data given $f = 5$

$$\tau_d = \frac{1}{5} = 0.2 \text{ sec.}$$

But $\tau_d = \frac{2\pi}{\omega_d}$ $\Rightarrow \omega_d = \frac{2\pi}{0.2} = 31.416$ rad/s

logarithmic decrement $\delta = \frac{1}{n} \ln\left(\frac{x_0}{x_n}\right)$

$$\delta = \frac{1}{60} \ln(0.15) = 0.0451$$

(b) damping ratio (ζ)

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} = \frac{0.0451}{\sqrt{(2\pi)^2 + 0.0451^2}} = 0.007177$$

(Ans)

Example - 2 . A single dof spring mass damper has a mass of 60 kg and a spring stiffness of 6000 N/m. Determine the following

(a) critical damping coefficient

(b) damped natural frequency when $c = 2c_c/3$

(c) logarithmic decrement.

(a) ~~$c_c = 2$~~ $m = 60$ kg. $K = 6000$ N/m.

$$c_c = 2m\omega_n = 2m \cdot \sqrt{\frac{K}{m}} = 2\sqrt{Km}.$$

$$= 2 \times \sqrt{6000 \times 60} = 1200 \text{ Ns/m}.$$

(b) Now $c = 2 \cdot \frac{c_c}{3} = 800$ Ns/m,

Damped natural frequency

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = \sqrt{\frac{K}{m}} \sqrt{1 - \left(\frac{c}{c_c}\right)^2}$$

$$= \sqrt{\frac{6000}{60}} \sqrt{1 - \left(\frac{800}{1200}\right)^2}$$

$$= 7.45 \text{ rad/sec}.$$

(c) Logarithmic decrement

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi\left(\frac{2}{3}\right)}{\sqrt{2\pi - (2/3)^2}}$$

$$= \boxed{5.6198}$$

Question - 1

A damper offers resistance of 0.05N at constant velocity 0.04 m/s. The damper is used with $K = 9 N/m$. Determine the damping and frequency of the ~~mass of the~~ system if the mass is 0.1 Kg.

We have Damping force $F = C\dot{x}$

$\dot{x} = 0.04\ m/s$. $F = 0.05N$.

$$C = \frac{F}{\dot{x}} = \frac{0.05}{0.04} = 1.25\ Ns/m.$$

$$C_c = 2\sqrt{Km} = 2 \times \sqrt{9 \times 0.1} = 1.897\ Ns/m.$$

damping factor $\xi = \frac{C}{C_c} = \frac{1.25}{1.897} = 0.658$

So the system is under damped

$$\omega_d = \omega_n\sqrt{1-\xi^2} = \sqrt{\frac{K}{m}}\sqrt{1-0.658^2}$$

$$= \sqrt{\frac{9}{0.1}}\sqrt{1-0.658^2}$$

Q-2 A vibrating system is defined by the following parameters:

$m = 3Kg$, $K = 100 N/m$, $C = 3 Ns/m$.

Determine (a) damping factor, (b) natural frequency of damped vibration (c) logarithmic decrement (d) ratio of two consecutive amplitudes (e) no of cycles after which the original amplitude is reduced to 20%.

Different types of Damping:-

The damping is a physical system may be ~~done~~ one of the several types.

1. Viscous damping:-

- It is one of the most important type of damping and occurs for small velocities in ~~lubricating~~ lubricated sliding surfaces, dashpots, with small clearances. The amount of damping resistance will depend upon the relative velocity and upon the parameters of the damping system.
- One of the reasons for so much importance of this type of damping is that it affords an easy analysis of system by virtue of the fact that differential equation for the system become linear with this type of damping.

2. Dry friction or Coulomb damping:-

This type of damping occurs when two machine parts rub against each other, dry or unlubricated. The damping resistance in this case is practically constant and it's independent of the rubbing velocity.

3. Solid or structural damping:-

This type of damping is due to the internal friction of the molecules. The stress-strain diagram for a vibrating body is not a straight line but forms a hysterisis loop, the area of which represents the energy dissipated due to molecular friction per cycle per unit volume. The size of the loop depends upon the material of the vibrating body, frequency and amount of dynamic stress.

4. Slip or interfacial damping:-

Energy of vibration is dissipated by microscopic slip on the interfaces of m/c parts in contact under fluctuating loads. Microscopic slips also occurs on the interfaces of m/c elements forming various types of joints. The amount of damping depends amongst other things upon the surface roughness of the mating parts, the contact pressure and amplitude of vibration. It is a non-linear type damping.

Equations of free damped single dof system

Solution of the equation

$$\boxed{s_{1,2} = \left(-\frac{c}{2m}\right) \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}}$$

Most general form of solution

$$\boxed{x = c_1 e^{s_1 t} + c_2 e^{s_2 t}}$$

Where c_1 and c_2 are two arbitrary constants to be determined from the initial conditions.

- ~~A term critical damping coefficient, denoted by c_c~~
~~It is that value of the damping coefficient c that makes the expression~~ $\cancel{\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}}$ ~~equals to zero~~

(i) over damped system $(\zeta > 1)$

$$s_1 = \left[-\zeta + \sqrt{\zeta^2 - 1}\right] \omega_n$$

$$s_2 = \left[-\zeta - \sqrt{\zeta^2 - 1}\right] \omega_n$$

Equation $$\boxed{x = c_1 e^{\left[-\zeta + \sqrt{\zeta^2 - 1}\right]\omega_n t} + c_2 e^{\left[-\zeta - \sqrt{\zeta^2 - 1}\right]\omega_n t}}$$

2. critically damped system ($\xi = 1$)

Roots $s_1 = s_2 = -\omega_n$

and. equation $\boxed{x = [c_1 + c_2 t]\, e^{-\omega_n t}}$

3. Underdamped system ($\xi < 1$)

$$\boxed{x = X e^{-\xi \omega_n t} \sin(\omega_d t + \phi)}$$

Assignment - 7 (P-73, Groover)

The mass of a spring mass dash pot system is given an initial velocity (from the equilibrium position) of $A\omega_n$. where ω_n is the undamped natural frequency of the system. find the equation of motion of the system for cases when (i) $\xi = 2$, (ii) $\xi = 1$, (iii) $\xi = 0.2$.

Assignment - 8

The disc of a torsional pendulum has a moment of inertia of 600 kg cm² and is immersed in a viscous fluid. The brass-shaft attached to it is of 10 cm dia and 40 cm long. When the pendulum is vibrating, it is observed amplitudes on the same side of the rest position for successive cycles are 9°, 6°, 4°. Determine

(a) logarithmic decrement

(b) damping torque at unit velocity

(c) periodic time of vibration.

Forced Vibration ~~with~~ of Single Degree of Freedom Systems

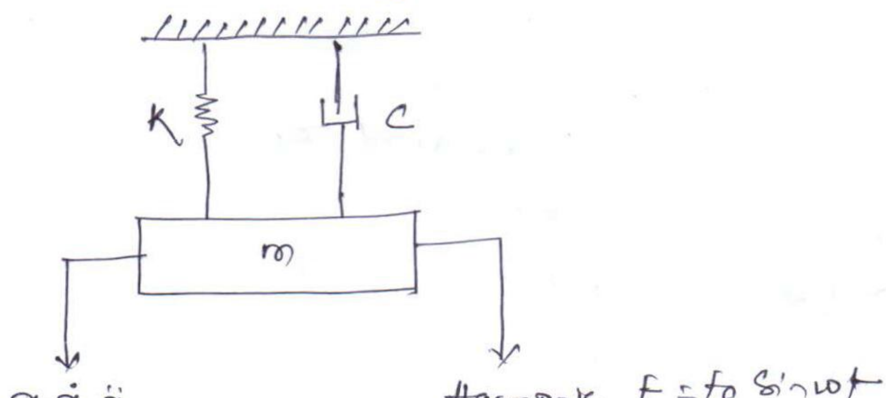
- In free vibration system, a system once disturbed from its equilibrium position, executes vibration because of it's elastic properties. The system will come to rest depending upon it's damping characteristics.
- In case of forced vibration there is an impressed force on the system which keeps it vibrating.

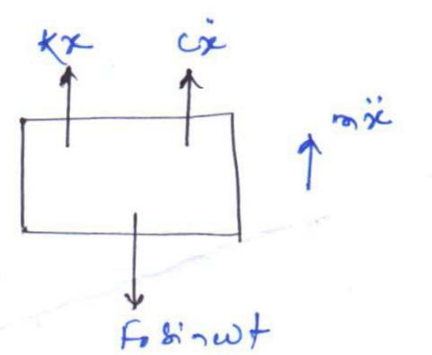
Examples:-

(a) air compressors
(b) Internal combustion engine
(c) machine tools and various other machineries.

Forced Vibration with Constant Harmonic Excitations:-

- In forced vibration the response of the system consists of two parts
  1. Transient and the system will vibrate with damped frequency
  2. steady state and the system will vibrate with the frequency of excitation.





From Newton's second law:

$$f_0 \sin \omega t - c\dot{x} - Kx - m\ddot{x} = 0$$

$$\Rightarrow m\ddot{x} + c\dot{x} + Kx = f_0 \sin \omega t \quad \text{---} (1)$$

Eq.(1) is a linear, second order differential equation and the solution has two parts.