2

Moment of inertia of planar cross-sections

2.1 Centre of Gravity

Every particle of a body is attracted by the earth towards its centre. The force of attraction, which is proportional to the mass of the particle, acts vertically downwards and is known as weight of the body.

A point may be found out in a body, through which the resultant of all such parallel forces act. This point, through which the whole weight of the body acts, irrespect of its position, is known as *centre of gravity* (briefly written as C.G.). It may be noted that every body has one and only one centre of gravity.

2.1.1 Centroid

The plane figures (like triangle, quadrilateral, circle etc.) have only areas, but no mass. The centre of area of such figures is known as centroid. The method of finding out the centroid of a figure is the same as that of finding out the centre of gravity of a body. In many books, centre of gravity for centroid and vice-versa.

2.1.2 Methods for Centre of Gravity

The centre of gravity (or centroid) may be found out by any one of the following two methods:

1. By geometrical considerations
2. By moments
3. By graphical method

Centre of Gravity by Geometrical Considerations

The centre of gravity of simple figures may be found out from the geometry of the figure as given below.

i. The centre of gravity (C.G.) of a uniform rod lies at its middle point.
ii. The centre of gravity of a triangle lies at the point where the three medians (a medians is a line connecting the vertex and the middle point of the opposite side) of the triangle meet.
iii. The centre of gravity of a rectangle or of a parallelogram is at the point, where its diagonal meet each other. It is also the point of interaction of the lines joining the middle points of the opposite sides.
iv. The centre of gravity of a circle is at its centre.

Centre of Gravity by Moments

The centre of gravity of a body may also be found out by moments as discussed below:
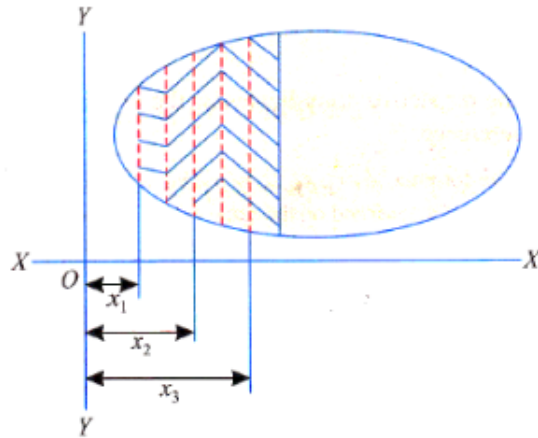
Fig.2.1 – Centre of gravity by moments

Consider a body of mass M whose centre of gravity is required to be found out. Divide the body into small masses, whose centres of gravity are known as shown in **Error! Reference source not found.**. Let m_1, m_2, m_3... etc. be the masses of the particles and (x_1, y_1), (x_2, y_2), (x_3, y_3),...... be the co-ordinates of the centres of gravity from a fixed point O as shown in .

Let $\bar{x}$ and $\bar{y}$ be the co-ordinates of the centre of gravity of the body. From the principle of moments, we know that

$$M\bar{x} = m_1 x_1 + m_2 x_2 + m_3 x_3 \text{ ...}$$

$$\bar{x} = \frac{\Sigma mx}{M}$$

$$\bar{y} = \frac{\Sigma my}{M}$$

Where, $\qquad M = m_1 + m_2 + m_3$

Axis of Reference

The centre of gravity of a body is always calculated with reference to some assumed axis known as axis of reference (or sometimes with reference to some point of reference). The axis of reference, of plane figures, is generally taken as the lowest line of the figure for calculating $\bar{y}$, and the left line of the figure for calculating $\bar{x}$.

Centre of Gravity of Plane Figures

The plane geometrical figures (such as T-section, I-section, L-section etc.) have only areas but no mass. The centre of gravity of such figures is found out in the same way as that of solid bodies. The centre of area of such figures is known as centroid, and coincides with the centre of gravity of the figure. It is a common practice to use centre of gravity for centroid and vice versa.

Let $\bar{x}$ and $\bar{y}$ be the co-ordinates of the centre of gravity with respect to some axis of reference, then

$$\bar{x} = \frac{a_1 x_1 + a_2 x_2 + a_3 x_3 + \dots}{a_1 + a_2 + a_3}$$

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3 + \dots}{a_1 + a_2 + a_3}$$

where a_1, a_2, a_3........ etc., are the areas into which the whole figure is divided x_1 x_2, x_3..... etc., are the respective co-ordinates of the areas a_1, a_2, a_3........ on X-X axis with respect to same axis of reference.

y_1 y_2, y_3..... etc., are the respective co-ordinates of the areas at, a_1, a_2, a_3........ etc., on Y-Y axis with respect to same axis of the reference.

Centre of Gravity of Symmetrical Sections

Sometimes, the given section, whose centre of gravity is required to be found out, is symmetrical about X-X axis or Y-Y axis. In such cases, the procedure for calculating the centre of gravity of the body is very much simplified; as we have only to calculate either $\bar{x}$ or $\bar{y}$. This is due to the reason that the centre of gravity of the body will lie on the axis of symmetry.

Ex. 2.1   *[Ex 9.1; R. S. Khurmi]*

Find the centre of gravity of a 100 mm x 150 x 30 mm T-section.

Solution: As the section is symmetrical about Y-Y axis, bisecting the web, therefore its centre of gravity will lie on this axis. Split up the section into two rectangles ABCH and DEFG as shown in below Fig.

Let bottom of the web FE be the axis of reference.

(i).   Rectangle ABCH

$$a_1 = 100 \times 30 = 3000 \ mm^2$$

$$y_1 = \left(150 - \frac{30}{2}\right) = 135 \ mm$$

(ii).   Rectangle DEFG

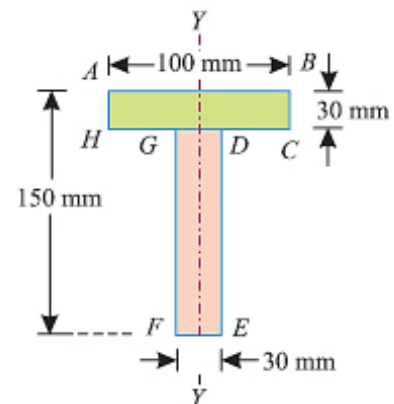$$a_2 = 120 \times 30 = 3600 \ mm^2$$

$$y_2 = \frac{120}{2} = 60 \ mm$$

The Distance between centre of gravity of the section and bottom of the flange FE,

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{(3000 \times 135) + (3600 \times 60)}{3000 + 3600}$$

$$\bar{y} = 94.1 \ mm$$

Ex. 2.2   *[Ex 9.2; R. S. Khurmi]*

Find the centre of gravity of a channel section 100 mm x 50 x 15 mm.

Solution: As the section is symmetrical about X-X axis, therefore its centre of gravity will lie on this axis. Now split up the whole section into three rectangles ABFJ, EGKJ and CDHK as shown in below Fig.

Let the face AC be the axis of reference.

(i).   Rectangle ABFJ

$$a_1 = 50 \times 15 = 750 \ mm^2$$

$$x_1 = \frac{50}{2} = 25 \ mm$$

(ii).   Rectangle EGKJ

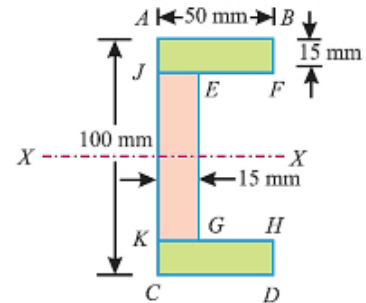$$a_2 = (100 - 30) \times 15 = 1050 \ mm^2$$

$$x_2 = \frac{15}{2} = 7.5 \ mm$$

(iii).   Rectangle CDHK

$$a_3 = 50 \times 15 = 750 \ mm^2$$

$$x_3 = \frac{50}{2} = 25 \ mm$$

The Distance between centre of gravity of the section and left face of the section AC,

$$\bar{x} = \frac{a_1 x_1 + a_2 x_2 + a_3 x_3}{a_1 + a_2 + a_3} = \frac{(750 \times 25) + (1050 \times 7.5) + (750 \times 25)}{750 + 1050 + 750}$$

$$\bar{x} = 17.8 \ mm$$

[Ex 9.3; R. S. Khurmi]

An I-section has the following dimensions in mm units :

Bottom flange = 300 × 100

Top flange = 150 × 50

Web = 300 × 50

Determine mathematically the position of centre of gravity of the section.

Solution: As the section is symmetrical about Y-Y axis, bisecting the web, therefore its centre of gravity will lie on this axis. Now split up the section into three rectangles as shown in Fig.

Let bottom of the bottom flange be the axis of reference.

(i). Bottom flange

$$a_1 = 300 \times 100 = 30000 \ mm^2$$

$$y_1 = \frac{100}{2} = 50 \ mm$$

(ii). Web

$$a_2 = 300 \times 50 = 15000 \ mm^2$$

$$y_2 = 100 + \frac{300}{2} = 250 \ mm$$
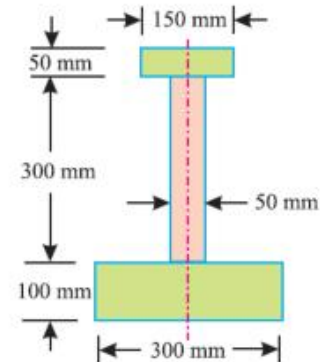
(iii). Top flange

$$a_3 = 150 \times 50 = 75000 \ mm^2$$

$$y_3 = 100 + 300 + \frac{50}{2} = 425 \ mm$$

The distance between centre of gravity of the section and bottom of the flange

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{a_1 + a_2 + a_3}$$

$$\bar{y} = \frac{(30000 \times 50) + (15000 \times 250) + (7500 \times 425)}{30000 + 15000 + 7500}$$

$$\bar{y} = 160.7 \ mm$$

Centre of Gravity of Unsymmetrical Sections

Sometimes, the given section, whose centre of gravity is required to be found out, is not symmetrical either about X-X axis or Y-Y axis. In such cases, we have to find out both the values of $\bar{x}$ and $\bar{y}$.

Ex. 2.4 **[Ex 9.4; R. S. Khurmi]**

Find the centroid of an unequal angle section 100 mm x 80 x 20 mm.

Solution: As the section is not symmetrical about any axis, therefore we have to find out the values of $\bar{x}$ and $\bar{y}$ for the angle section. Split up the section into two rectangles as shown in below Fig.

Let left face of the vertical section and bottom face of the horizontal section be axes of reference.

(i). **Rectangle 1**

$$a_1 = 100 \times 20 = 2000 \ mm^2$$

$$x_1 = \frac{20}{2} = 10 \ mm$$

$$y_1 = \frac{100}{2} = 50 \ mm$$

(ii). **Rectangle 2**

$$a_2 = (80 - 20) \times 20 = 1200 \ mm^2$$

$$x_2 = 20 + \frac{60}{2} = 50 \ mm$$

$$y_2 = \frac{20}{2} = 10 \ mm$$

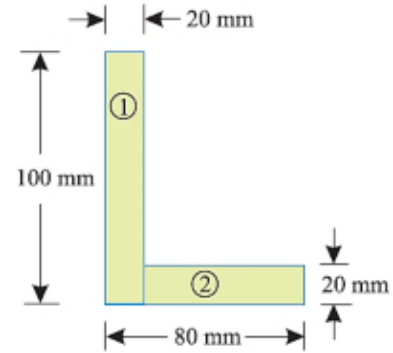The Distance between centre of gravity of the section and left face,

$$\bar{x} = \frac{a_1 x_1 + a_2 x_2}{a_1 + a_2} = \frac{(2000 \times 10) + (1200 \times 50)}{2000 + 1200}$$

$$\bar{x} = 25 \ mm$$

The Distance between centre of gravity of the section and bottom face,

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{(2000 \times 50) + (1200 \times 10)}{2000 + 1200}$$

$$\bar{y} = 35 \ mm$$

2.2 Moment of Inertia

The moment of a force (P) about a point, is the product of the force and perpendicular distance (x) between the point and the line of action of the force (i.e. P.x). This moment is also called first moment of force. If this moment is again multiplied by the perpendicular distance (x) between the point and the line of action of the force i.e. P.x (x) = Px^2, then this quantity is called moment of the moment of a force or second moment of force or **Moment of Inertia (M.I.).**

Sometimes, instead of force, area or mass of a figure or body is taken into consideration. Then the second moment is known as second moment of area or second moment of mass. But all such second moments are broadly termed as moment of inertia.

Moment of Inertia of A Plane Area

Consider a plane area, whose moment of inertia is required to be found out. Split up the whole area into a number of small elements.

Let a_1, a_2, a_3, ... = Areas of small elements, and

r_1, r_2, r_3, ... = Corresponding distances of the elements from the line about which the moment of inertia is required to be found out.

Now the moment of inertia of the area,

$$I = a_1 r_1^2 + a_2 r_2^2 + a_3 r_3^2 + \cdots$$

$$I = \Sigma \, ar^2$$

As a matter of fact the units of moment of inertia of a plane area depend upon the units of the area and the length. e.g.,

1. If area is in m^2 and the length is also in m, the moment of inertia is expressed in m^4.
2. If area in mm^2 and the length is also in mm, then moment of inertia is expressed in mm^4.

2.2.1 Theorem of Perpendicular Axis

It states, If I_{xx} and I_{yy} be the moments of inertia of a plane section about two perpendicular axis meeting at O, the moment of inertia I_{zz} about the axis Z-Z, perpendicular to the plane and passing through the intersection of X-X and Y-Y is given by:

$$I_{zz} = I_{xx} + I_{yy}$$

Proof

Consider a small lamina (P) of area *da* having co-ordinates as x and y along OX and OY two mutually perpendicular axes on a plane section as shown in *Fig.2.1Fig.2.2.*
Now consider a plane OZ perpendicular to OX and OY. Let (r) be the distance of the lamina (P) from Z-Z axis such that OP = r.
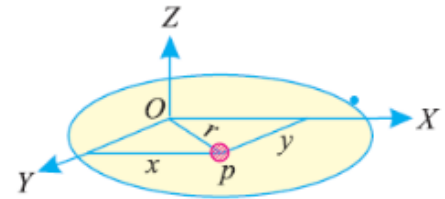

Fig.2.2 – Theorem of perpendicular axis.

From the geometry of the figure, we find that

$$r^2 = x^2 + y^2$$

The moment of inertia of the lamina P about X-X axis,

$$I_{xx} = da.\, y^2 \qquad\qquad [I = Area \times (Distance)^2]$$

Similarly, $\qquad I_{yy} = da.\, x^2$

$$I_{zz} = da.\, r^2 = da\,(x^2 + y^2) \qquad\qquad (r^2 = x^2 + y^2)$$
$$= da.\, x^2 + da.\, y^2$$
$$= I_{YY} + I_{xx}$$

2.2.2 Theorem of Parallel Axis

It states, If the moment of inertia of a plane area about an axis through its centre of gravity is denoted by I_G, then moment of inertia of the area about any other axis AB, parallel to the first, and at a distance h from the centre of gravity is given by:

$$I_{AB} = I_G + ah^2$$

where I_{AB} = Moment of inertia of the area about an axis AB,

$\quad I_G$ = Moment of Inertia of the area about its centre of gravity

$\quad a$ = Area of the section, and

$\quad h$ = Distance between centre of gravity of the section and axis AB.

Proof

Consider a strip of a circle, whose moment of inertia is required to be found out about a line AB as shown in *Fig.2.3*.

$\quad$ Let δa = Area of the strip

$\quad\quad y$ = Distance of the strip from the centre of gravity the section

and

$\quad\quad h$ = Distance between centre of gravity of the section and the axis AB.
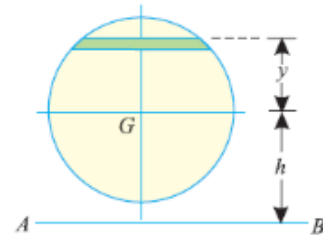

Fig.2.3 – Theorem of parallel axis

The moment of inertia of the whole strip about an axis passing through the centre of gravity of the section

$$= \delta a. y^2$$

Moment of inertia of the whole section about an axis passing through its centre of gravity,

$$I_G = \Sigma \, \delta a. y^2$$

Moment of inertia of the section about the axis AB,

$$I_{AB} = \Sigma \, \delta a. (h + y)^2$$
$$I_{AB} = \Sigma \, \delta a. (h^2 + y^2 + 2hy)$$
$$I_{AB} = \Sigma \, \delta a. h^2 + \Sigma \, \delta a. y^2 + \Sigma \, \delta a. 2hy)$$
$$I_{AB} = ah^2 + I_G + 0$$

$\Sigma \, \delta a.y$ is the algebraic sum of moments of all the areas, about an axis through centre of gravity of the section and is equal to $a.\bar{y}$, where $\bar{y}$ is the distance between the section and the axis passing through the centre of gravity, which obviously is zero.

$$\therefore I_{AB} = ah^2 + I_G$$

2.2.3 Moment of Inertia of a Rectangular Section

Consider a rectangular section ABCD as shown in *Fig.2.4* whose moment of inertia is required to be found out.

     Let b = Width of the section and

         d = Depth of the section.

Now consider a strip PQ of thickness dy parallel to X-X axis and at a distance y from it as shown in the figure

     ∴ Area of the strip = b.dy

We know that moment of inertia of the strip about X-X axis,

     = Area × y² = (b. dy) y²

     = b. y². dy

Moment of inertia of the whole section may be found out by integrating the above equation for the whole length of the lamina i.e. from −d/2 to +d/2,

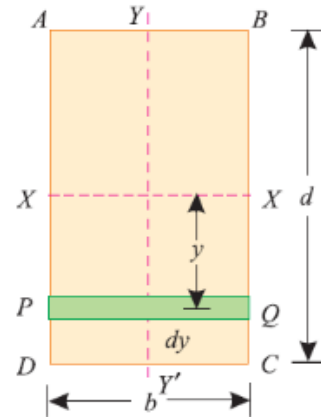*Fig.2.4 – Rectangular section.*

$$I_{XX} = \int_{-\frac{d}{2}}^{+\frac{d}{2}} b.y^2.dy = b \int_{-\frac{d}{2}}^{+\frac{d}{2}} y^2.dy$$

$$I_{XX} = b\left[\frac{y^3}{3}\right]_{-\frac{d}{2}}^{+\frac{d}{2}} = b\left[\frac{(d/2)^3}{3} - \frac{(-d/2)^3}{3}\right]$$

$$I_{XX} = \frac{bd^3}{12}$$

$$I_{YY} = \frac{bd^3}{12}$$

2.2.4 Moment of Inertia of a Circular Section

Consider a circle ABCD of radius (r) with centre O and X-X' and Y-Y' be two axes of reference through O as shown in *Fig.2.5*.

Now consider an elementary ring of radius x and thickness dx.

Therefore area of the ring,

$$da = 2\pi x.\,dx$$

Moment of inertia of ring, about X-X axis or Y-Y axis

$$= \text{area} \times (Distance)^2$$

$$= 2\pi x.\,dx \times (x)^2$$

$$= 2\pi\, x^3.\,dx$$

Moment of inertia of the whole section, about the central axis, can be found out by integrating the above equation for the whole radius of the circle i.e., from 0 to r.

Fig.2.5 – Circular section.

$$I_{ZZ} = \int_0^r 2\pi\, x^3.\,dx = 2\pi \int_0^r x^3.\,dx$$

$$I_{ZZ} = 2\pi \left[\frac{x^4}{4}\right]_0^r = \frac{\pi}{2} r^4$$

$$I_{ZZ} = \frac{\pi}{32} d^4$$

According to the Theorem of Perpendicular Axis that

$$I_{ZZ} = I_{XX} + I_{YY}$$

$$I_{XX} = I_{YY} = \frac{I_{ZZ}}{2} = \frac{1}{2} \times \frac{\pi}{32} d^4$$

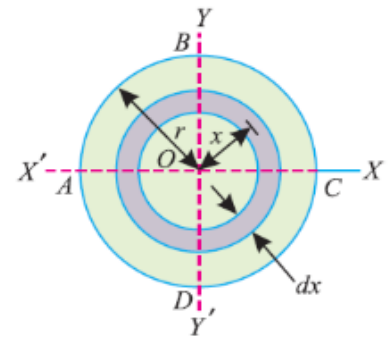$$I_{XX} = I_{YY} = \frac{\pi}{64} d^4$$

Ex. 2.5	[Ex 4.5; R. S. Khurmi]
	Fig.2.6 shows an area ABCDEF. Compute the moment of inertia of the above area about axis K-K.

Solution: As the moment of inertia is required to be found out about the axis K-K, therefore there is no need of finding out the centre of gravity of the area.

Let us split up the area into two rectangles 1 and 2.

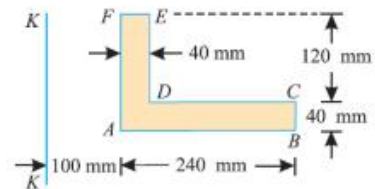Moment of inertia of section (1) about its centre of gravity and parallel to axis K-K,

Fig.2.6

$$I_{G1} = \frac{120 \times (40)^3}{12}$$

$$I_{G1} = 640 \times 10^3 \ mm^4$$

Distance between centre of gravity of section (1) and axis K-K,

$$h_1 = 100 + \frac{40}{2} = 120 \text{ mm}$$

Moment of inertia of section (1) about axis K-K

$$= I_{G1} + a_1 h_1^2$$

$$= (640 \times 10^3) + [(120 \times 40) \times (120)^2]$$

$$= 69.76 \times 10^6 \ mm^4$$

Moment of inertia of section (2) about its centre of gravity and parallel to axis K-K,

$$I_{G2} = \frac{40 \times (240)^3}{12}$$

$$I_{G2} = 46.08 \times 10^6 \ mm^4$$

Distance between centre of gravity of section (2) and axis K-K,

$$h_2 = 100 + \frac{240}{2} = 220 \ mm$$

Moment of inertia of section (2) about axis K-K

$$= I_{G2} + a_2 h_2^2$$

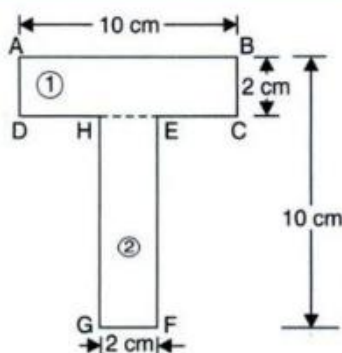$$= (46.08 \times 10^6) + [(240 \times 40) \times (220)^2]$$

$$= 510.72 \times 10^6 \ mm^4$$

Moment of inertia of the whole area about axis K-K,

$$I_{KK} = (69.76 \times 10^6) + (510.72 \times 10^6)$$

$$\boldsymbol{I_{KK} = 580.48 \times 10^6 \ mm^4}$$

Ex. 2.6	[Ex 5.14; R. K. Bansal]

Below Figure shows a T-section of dimensions 10 x 10 x 2 cm. Determine the moment of inertia of the section about the horizontal and vertical axes, passing through the centre of gravity of the section.

Solution:

The given section is symmetrical about the axis Y-Y and hence the C.G. of the section will lie on Y-Y axis. The given section is split up into two rectangles ABCD and EFGH for calculating the C.G. of the section.

Area of rectangle ABCD,

$a_1 = 10 \times 2 = 20 \ cm^2$

Distance of C.G. of the area ax from the bottom line GF,

$y_1 = 8 + 1 = 9 \ cm$

Area of rectangle EFGH, $a_2 = 6 \times 2 = 16 \ cm^2$

Distance of C.G. of rectangle EFGH from the bottom line GF, $y_2 = 4 \ cm$

Distance of the **C.G.** of the section from the bottom line GF

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2}$$

$$\bar{y} = \frac{(20 \times 9) + (16 \times 4)}{20 + 16}$$

$$\bar{y} = 6.77 \; cm$$

The distance between the C.G. of the given section and the C.G. of the rectangle (1)

$h_1 = y_1 - \bar{y} = 9 - 6.67 = 2.23 \; cm$

The distance between the C.G. of the given section and the C.G. of the rectangle (2)

$h_2 = \bar{y} - y_2 = 6.67 - 4 = 2.77 \; cm$

Moment of inertia of rectangle (1) about the horizontal axis and passing through its C.G.

$$I_{G_1} = \frac{10 \times 2^3}{12} = 6.67 \; cm^4$$

Moment of inertia of rectangle (2) about the horizontal axis and passing through the C.G. of the rectangle (2)

$$I_{G_2} = \frac{2 \times 8^3}{12} = 85.33 \; cm^4$$

From the theorem of parallel axes, the moment of inertia of the rectangle (1) about the horizontal axis passing through the C.G. of the given section

$$= I_{G_1} + a_1 h_1^2 = 6.67 + 20 \times (2.23)^2$$

$$= 105.501 \; cm^4$$

The moment of inertia of the rectangle (2) about the horizontal axis passing through the C.G. of the given section

$$= I_{G_2} + a_2 h_2^2 = 85.33 + 16 \times (6.67)^2$$

$$= 208.72 \; cm^4$$

The moment of inertia of the given section about the horizontal axis passing through the C.G. of the given section is,

$$I_{xx} = 105.501 + 208.72$$

$$= 314.221 \; cm^4$$

The moment of inertia of the given section about the vertical axis passing through the C.G. of the given section is,

$$I_{yy} = \frac{2 \times 10^3}{12} + \frac{8 \times 2^3}{12}$$

$$= 172 \; cm^4$$

Now the polar moment of inertia

$$I_{zz} = I_{xx} + I_{yy}$$

$$= 314.221 + 172$$

$$= 486.221 \; cm^4$$

Ex. 2.7 *[Ex 5.15; R. K. Bansal]*

Find the moment o f inertia of the section shown in below figure about the centroidal axis X-X perpendicular to the web.

Solution:
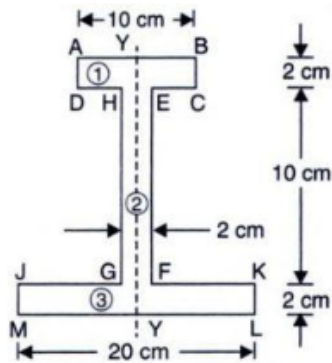
The given section is symmetrical about the axis Y-Y and hence the C.G. of the section will lie on Y- Y axis. The given section is split up into three rectangles ABCD, EFGH and JKLM.

Area of rectangle ABCD,
$a_1 = 10 \times 2 = 20$ cm²
Distance of C.G. of the rectangle ABCD from the bottom line ML,
$y_1 = 2 + 10 + 1 = 13$ cm
Area of rectangle EFGH,
$a_2 = 10 \times 2 = 20$ cm²

Distance of C.G. of rectangle EFGH from the bottom line ML,
$y_2 = 2 + 5 = 7$ cm
Area of rectangle JKLM,
$a_3 = 20 \times 2 = 40$ cm²
Distance of C.G. of rectangle EFGH from the bottom line ML,
$y_3 = 1$ cm
The centre of gravity of the section is obtained by using

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{a_1 + a_2 + a_3}$$

$$\bar{y} = \frac{(20 \times 13) + (20 \times 7) + (40 \times 1)}{20 + 20 + 40}$$

$$\bar{y} = 5.50 \ cm$$

The distance between the C.G. of the given section and the C.G. of the rectangle (1)
$h_1 = y_1 - \bar{y} = 13 - 5.5 = 7.5 \ cm$
The distance between the C.G. o f the given section and the C.G. o f the rectangle (2)
$h_2 = y_2 - \bar{y} = 7 - 5.5 = 1.5 \ cm$
The distance between the C.G. o f the given section and the C.G. o f the rectangle (3)
$h_2 = \bar{y} - y_3 = 5.5 - 1 = 4.5 \ cm$

Moment of inertia of rectangle (1) about the horizontal axis and passing through its C.G.

$$I_{G_1} = \frac{10 \times 2^3}{12} = 6.67 \ cm^4$$

Moment o f inertia o f rectangle (2) about the horizontal axis and passing through the C.G. of the rectangle (2)

$$I_{G_2} = \frac{2 \times 10^3}{12} = 166.67 \ cm^4$$

Moment o f inertia o f rectangle (3) about the horizontal axis and passing through the C.G. of the rectangle (3)

$$I_{G_3} = \frac{20 \times 2^3}{12} = 13.33 \ cm^4$$

From the theorem of parallel axes, the moment of inertia of the rectangle (1) about the horizontal axis passing through the C.G. of the given section

$$= I_{G_1} + a_1 h_1^2 = 6.67 + 20 \times (7.5)^2$$

$$= 1131.67 \ cm^4$$

The moment of inertia of the rectangle (2) about the horizontal axis passing through the C.G. of the given section

$$= I_{G_2} + a_2 h_2^2 = 166.67 + 20 \times (1.5)^2$$

$$= 211.67 \ cm^4$$

The moment of inertia of the rectangle (3) about the horizontal axis passing through the C.G. of the given section

$$= I_{G_3} + a_3 h_3^2 = 13.33 + 40 \times (4.5)^2$$

$$= 823.33 \ cm^4$$

The moment of inertia of the given section about the horizontal axis passing through the C.G. of the given section is,

$$= 1131.667 + 211.667 + 823.33$$

$$= \mathbf{2166.67 \ cm^4}$$