પાના નં.:
તારીખ: / /

* Work done in stretching a wire :

- Work is given by

$W = \vec{F} \cdot \vec{d}$

$= Fd \cos\theta$

Here,

F = applied force, and
d = displacement of the object on application of applied force, and
θ = angle between the applied force and the displacement.

- When a force, along the length, is applied on a wire, its length increases.

- If the original length of the wire is L and the increase in length (the stretch) of the wire is 'l', then the work done in stretching a wire by a very small amount 'dl' is given by

$dW = F \cdot dl$ ——— ①

Here, $\cos\theta = 1$, because force applied on the wire is parallel to its length.

- Therefore, the total work done in stretching a wire by an amount 'l' is obtained as follows.

$$W = \int dW = \int_0^l F \, dl \quad \text{——— (2)}$$

- In above equation force F can be obtained using the equation of Young's modulus

- Therefore,

$$\therefore Y = \frac{F/A}{l/L} = \frac{F}{A} \times \frac{L}{l}$$

$$\therefore F = \frac{YAl}{L} \quad \text{——— (3)}$$

- Substituting the value of eqn (3) into eqn (2),

$$W = \int_0^l \left(\frac{YAl}{L}\right) dl$$

$$= \frac{YA}{L} \int_0^l l \, dl$$

$$= \frac{YA}{L} \left[\frac{l^2}{2}\right]_0^l$$

$$\therefore W = \frac{1}{2} \cdot \left[\frac{YAl}{L}\right] \cdot l$$

$$\therefore W = \frac{1}{2} F l \quad \text{——— (4)}$$

$$\therefore W = \frac{1}{2} \times \text{stretching Force} \times \text{Stretching} \quad \text{——— (5)}$$

- Now, let us divide equation ④ by the volume of the wire.

$$\therefore \frac{W}{V} = \frac{1}{2}\frac{Fl}{V}$$

$$= \frac{1}{2} \times \frac{F}{A} \times \frac{l}{L} \qquad \{\because V = A \cdot L$$

$$\therefore \frac{W}{V} = \frac{1}{2} \times \text{Stress} \times \text{Strain} \qquad \text{——— ⑥}$$

- Here, equations ④ & ⑤ represent work done in stretching a wire while equation ⑥ represents the work done in stretching a wire per unit volume (if the volume of the wire is taken as unity, i.e. $V = 1\ m^3$)

* Twisting couple on a Cylinder or a wire or shaft.

- Consider a solid cylinder of length L and radius R clamped at its upper end twisted by a couple force applied to its lower end in a plane perpendicular to its length through an angle θ, as shown in fig. 2

- Due to the elasticity of the material of the cylinder, a restoring couple force is set up in the cylinder.

- To estimate this couple force, assume the solid cylinder consisting of a large number of hollow coaxial cylinders, one inside the other. One such cylinder of radius x and thickness dx is shown in fig.-1 & 2.

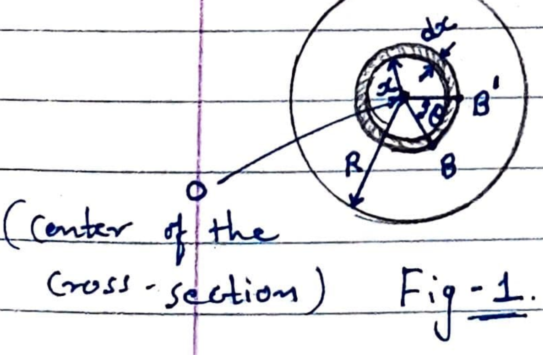


Fig-1.

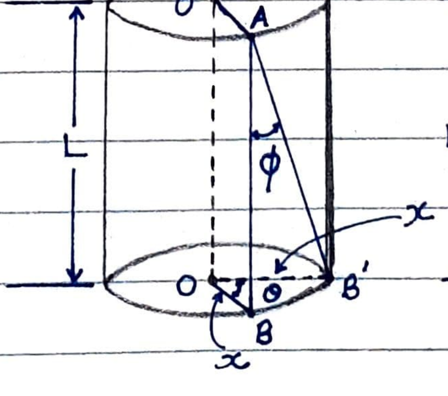


Fig-2

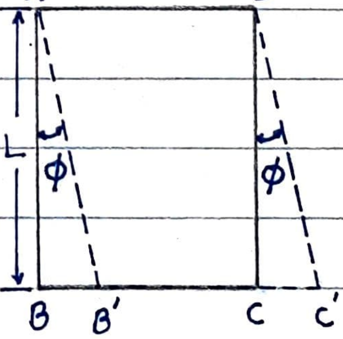


Fig-3

- In fig-2 a straigth line AB is parallel to the axis OO' of the cylinder (before the cylinder is been twisted at its lower end).
- When the cylinder is twisted at its lower end by an angle θ, the point B will move to the point B'.
- Now, the line AB' is making an angle with the line AB, this angle $\angle BAB' = \phi$ is called the angle of shear (or shear).

- Now, from fig-2 it is clear that the arc,

$$\overset{\frown}{BB'} = x\theta, \text{ and also}$$

$$\overset{\frown}{BB'} = L\phi.$$

$$\left\{ \because \text{Angle (in radians)} = \frac{\text{Arc}}{\text{Radius}} \right.$$

$$\therefore \quad x\theta = L\phi$$

$$\therefore \text{ shear... } \phi = \frac{x\theta}{L} \quad \text{——— (1)}$$

- Now, the modulus of rigidity of the material of the cylinder is given by

$$G = \frac{\text{Shearing stress}}{\text{Shearing strain (or shear)}}$$

$$\therefore \text{ shearing stress} = G \times \text{shear}$$

$$= G \times \phi \quad \text{——— (2)}$$

Now, substituting the value of eqn. ① into eqn. ② we get

$$\text{Shearing stress} = G\left(\frac{x\theta}{L}\right)$$

$$\therefore \frac{F}{dA} = \frac{Gx\theta}{L}$$

$\therefore$ shearing force ...

$$F = \left(\frac{Gx\theta}{L}\right) \cdot dA$$

$$= \left(\frac{Gx\theta}{L}\right) \cdot (2\pi x\, dx) \quad \left\{ \because dA = 2\pi x\, dx \right.$$

$$\therefore F = \left(\frac{2\pi G\theta}{L}\right) x^2\, dx \quad \text{——— ③}$$

- Now, the torque (moment of force) $d\tau$ at the lower end of the cylinder, about the axis OO' is given by

$$d\tau = F \cdot x$$

$$\therefore d\tau = \left(\frac{2\pi G\theta}{L}\right) x^2\, dx \cdot x$$

$$\therefore d\tau = \left(\frac{2\pi G\theta}{L}\right) x^3\, dx \quad \text{——— ④}$$

- Therefore, the twisting couple (torque) τ on the whole cylinder of radius R can be obtained by taking integral over the radius $x=0$ to $x=R$.

$$\therefore \tau = \int d\tau$$

$$\therefore \tau = \int_0^R \left(\frac{2\pi G \theta}{L}\right) x^3 \, dx$$

$$= \frac{2\pi G \theta}{L} \int_0^R x^3 \, dx$$

$$= \frac{2\pi G \theta}{L} \left[\frac{x^4}{4}\right]_0^R$$

$$\therefore \tau = \frac{\pi G \theta R^4}{2L} \quad \text{——— (5)}$$

- Twisting couple per unit twist (or torque per unit twist), i.e. τ/θ of the cylinder is given by

$$C_s = \frac{\tau}{\theta} = \frac{\pi G R^4}{2L} \quad \text{——— (6)}$$

(For solid cylinder)

- <u>Note</u>: τ/θ is also known as torsional rigidity C of the material of the cylinder (or wire).

* Torque on a Hollow cylinder

Torque on a hollow cylinder, whose inner radius is R_i and outer radius is R_o, can be obtained by taking integral over the radius $x = R_i$ to $x = R_o$ in equation ④.

$$\therefore \quad T = \int_{R_i}^{R_o} \left(\frac{2\pi G \theta}{L} \right) x^3 \, dx$$

$$\therefore \quad T = \frac{2\pi G \theta}{L} \int_{R_i}^{R_o} x^3 \, dx$$

$$\therefore \quad T = \frac{2\pi G \theta}{L} \left[\frac{x^4}{4} \right]_{R_i}^{R_o}$$

$$\therefore \quad T = \frac{\pi G \theta}{2L} \left(R_o^4 - R_i^4 \right) \quad \text{—— ⑦}$$

→ Therefore, torque per unit twist for a hollow cylinder is,

$$C_h = \frac{T}{\theta} = \frac{\pi G}{2L} \left(R_o^4 - R_i^4 \right) \quad \text{—— ⑧}$$

પાન નં :
તારીખ: / /

* Comparision of torsional rigidity of a solid and a hollow cylinder of same density, length and mass.

- From equations ⑥ & ⑧

$$\frac{C_h}{C_s} = \frac{(R_o^4 - R_i^4)}{R^4}$$

$$= \frac{(R_o^2 + R_i^2)(R_o^2 - R_i^2)}{R^4} \quad \text{—— (9)}$$

- Now, density is given by $\rho = \frac{m}{V}$.

- Volume of a solid cylinder... $V_s = \pi R^2 L_s$, and

- Volume of a hollow cylinder... $V_h = \pi(R_o^2 - R_i^2) L_h$

- Therefore, mass of a solid cylinder...

$$m_s = \rho_s \cdot V_s = \rho_s \cdot \pi R^2 L_s \quad \text{—— (10)}$$, and

mass of a hollow cylinder...

$$m_h = \rho_h \cdot V_h = \rho_h \cdot \pi (R_o^2 - R_i^2) L_h \quad \text{—— (11)}$$

- Now, for the cylinders of same mass, density and length

$$R^2 = R_o^2 - R_i^2 \quad \text{—— (12)} \quad \left\{ \text{From eqns (10) \& (11)} \right.$$

Therefore, from eqns. (9) & (12).

$$\frac{C_h}{C_s} = \frac{(R_o^2 + R_i^2) \cdot R^2}{R^4}$$

$$\therefore \frac{C_h}{C_s} = \frac{(R_o^2 + R_i^2)}{R^2} \qquad \text{——— (13)}$$

- Now, from eqn (12)

$$R^2 = R_o^2 - R_i^2$$

$$\therefore R_o^2 = (R^2 + R_i^2)$$

$$\therefore R_o^2 + R_i^2 = (R^2 + R_i^2) + R_i^2 \qquad \left\{ \because \text{Adding } R_i^2 \text{ on both the sides} \right.$$

$$\therefore R_o^2 + R_i^2 = R^2 + 2R_i^2$$

From the above equation it is clear that,

$$(R_o^2 + R_i^2) > R^2$$

- Therefore, from equation (13)

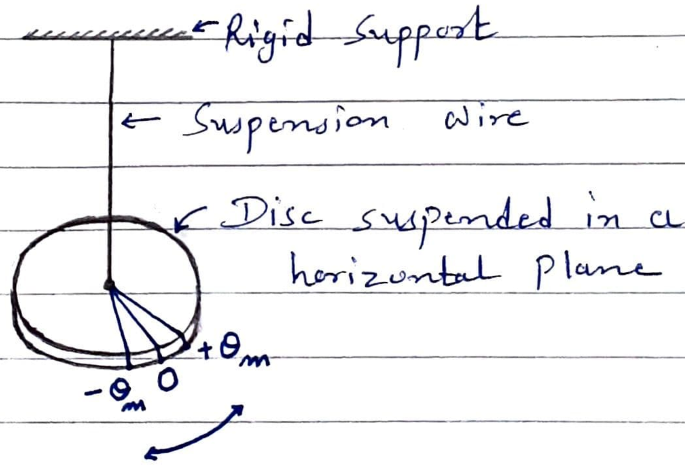
$$\frac{C_h}{C_s} = \frac{(R_o^2 + R_i^2)}{R^2} > 1$$

$$\therefore C_h > C_s \qquad \text{——— (14)}$$

- Above relation indicates that twisting couple per unit twist for a hollow cylinder is greater than that of a solid cylinder having same material, mass and length.

- This is the reason why hollow shafts are prefered over the solid ones for transmitting large torque in rotating machinery.

* Torsional Pendulum :



In the above figure a disc is suspended horizontally by a wire attached to its centre. This set-up is known as torsion pendulum.

- Here, the suspension wire is inextensible and free to be twisted about its axis.

- Due to the twist produced in the wire, the disc attached to it is rotated in the horizontal plane.

In the above figure θ represents the angle of rotation (or angular displacement) of the disc and $\theta = 0$ represents the untwisted or initial condition of the wire.

- Here, the restoring torque produced in the wire (due to its twisting) is given as

$$\tau \propto \theta$$

$$\therefore \tau = -C\theta \quad \text{——} \quad (1)$$

where, C = torsional rigidity of the material of the wire.

- Equation (1) is the torsional equivalent of Hooke's law $F = -ky$ (For simple harmonic motion).

- Now torque is also given by the eqn

$$\tau = I\alpha \quad \text{——} \quad (2)$$

where, I = moment of inertia of the disc about a perpendicular axis passing through its center.
α = angular acceleration.

- Therefore, from eqns (1) & (2),

$$I\alpha = -C\theta$$

$$\therefore I\alpha + C\theta = 0$$

$$\therefore \alpha + \frac{C}{I}\theta = 0$$

$$\therefore \frac{d^2\theta}{dt^2} + \frac{C}{I}\theta = 0$$

$$\therefore \frac{d^2\theta}{dt^2} + \omega^2\theta = 0 \quad \left\{ \because \frac{C}{I} = \omega^2 \right.$$

- Above equation is the differential equation of simple harmonic motion for torsional pendulum.

Now, $\omega^2 = \frac{C}{I}$

$$\therefore \omega = \sqrt{\frac{C}{I}}$$

$$\therefore \frac{2\pi}{T} = \sqrt{\frac{C}{I}}$$

$$\therefore T = 2\pi\sqrt{\frac{I}{C}} \quad \text{—— (3)}$$

- Equation (3) represents the time period of oscillation of torsional pendulum.