Forced Vibration of Single Degree of Freedom Systems

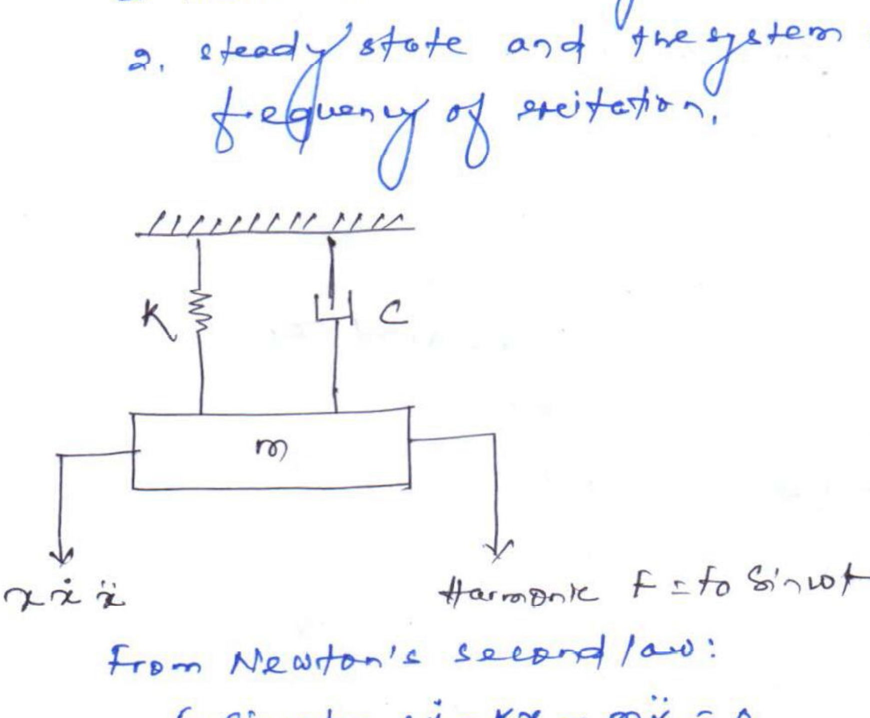
- In free vibration system, a system once disturbed from its equilibrium position, executes vibration because of it's elastic properties. The system will come to rest depending upon its damping characteristics.
- In case of forced vibration there is an impressed force on the system which keeps it vibrating.

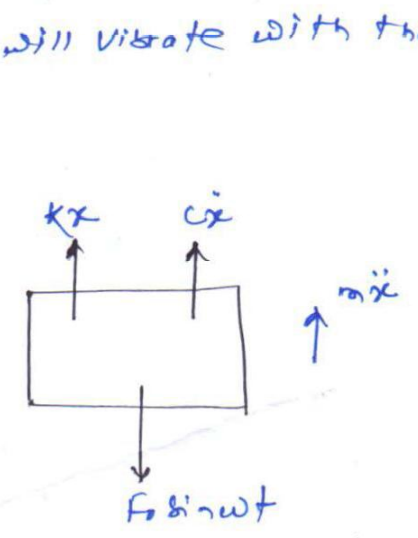
Examples:-

(a) air compressors
(b) Internal combustion engine
(c) machine tools and various other machineries.

Forced Vibration with Constant Harmonic Excitations:-

- In forced vibration the response of the system consists of two parts
  1. Transient and the system will vibrate with damped frequency
  2. steady state and the system will vibrate with the frequency of excitation.





From Newton's second law:

$$f_0 \sin\omega t - c\dot{x} - Kx - m\ddot{x} = 0$$

$$\Rightarrow m\ddot{x} + c\dot{x} + Kx = f_0 \sin\omega t \quad \text{---(1)}$$

Eq.(1) is a linear, second order differential equation and the solution has two parts.

- complementary function (transient part will dissapear)
- Particular integral

for complementary solution $m\ddot{x}+c\dot{x}+Kx=0$

- The particular solution is a steady state harmonic oscillation having a frequency equal to the excitation, and the displacement vector lags the force vector by some angle.

- Let the particular solution be $x_p = X\sin(\omega t-\phi)$ — (2)

where X = amplitude of vibration

$$\dot{x}_p = \omega X\cos(\omega t-\phi) = \omega X\sin(\omega t-\phi+\pi/2)$$

$$\ddot{x}_p = \omega^2 X\sin(\omega t-\phi+\pi)$$

As the complementary solution x_c will dissapear, we have

$$m\ddot{x}_p + c\dot{x}_p + Kx_p = f_0\sin\omega t$$

$$\Rightarrow f_0\sin\omega t - m\ddot{x}_p - c\dot{x}_p - Kx_p = 0 \quad \text{— (3)}$$

where $f_0\sin\omega t$ = impressed force

$m\ddot{x}_p$ = inertia force

Kx_p = spring force.

$c\dot{x}_p$ = damping force.

Substituting the values of $\ddot{x}_p$, $\dot{x}_p$ and x_p in eq.(3)

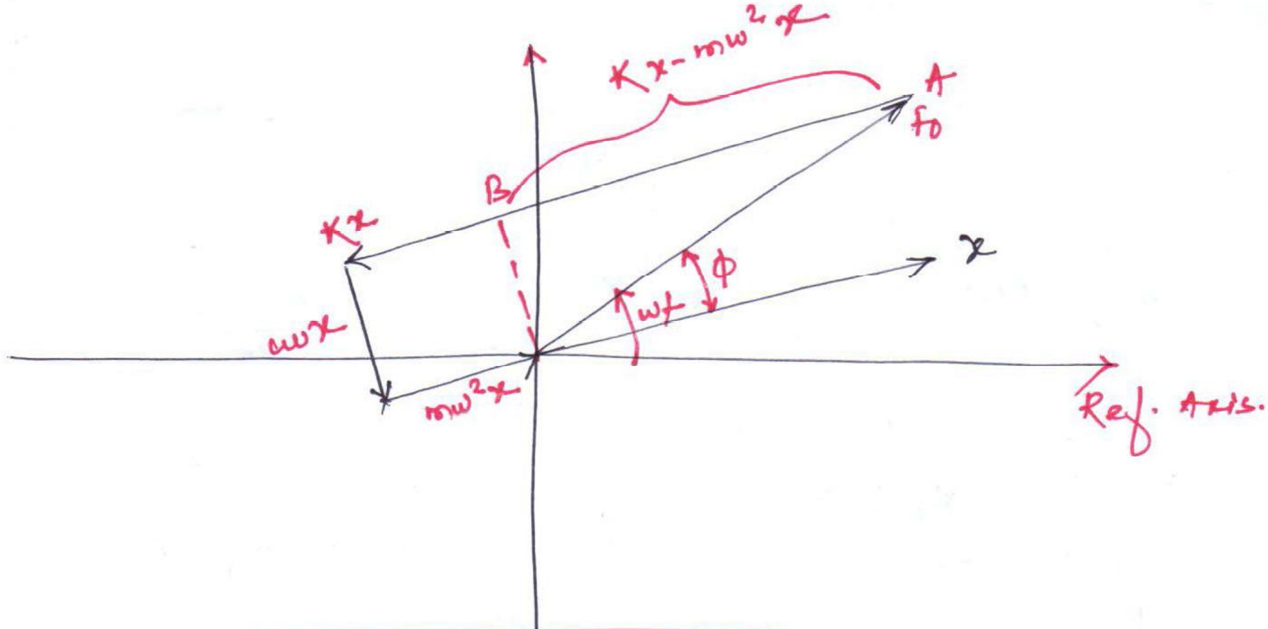
$$f_0\sin\omega t - m\omega^2X\sin(\omega t-\phi+\pi)$$
$$- c\omega X\sin(\omega t-\phi+\pi/2) - KX\sin(\omega t-\phi) = 0 \quad \text{— (4)}$$

The vectorial representation of equation (4) is as shown in the figure.

from the figure, we have $\tan\phi = \dfrac{c\omega X}{(KX - m\omega^2 X)}$

$$= \frac{c\omega}{(K-m\omega^2)} = \frac{c\omega/K}{\left(1-\frac{m\omega^2}{K}\right)}$$

$$= \frac{\frac{c}{c_c}\cdot\frac{c_c}{2m}\cdot\frac{2m\omega}{K}}{\left(1-\frac{\omega^2}{\omega_n^2}\right)} = \frac{\zeta\cdot\omega_n\cdot\frac{2\omega}{\omega_n^2}}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]} = \frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}$$



Now $\boxed{\phi = \tan^{-1} \dfrac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]}}$ — (5)

when
$\zeta = 0 \quad \frac{\omega}{\omega_n} < 1, \quad \phi = 0$
$\zeta = \text{any value}, \quad \frac{\omega}{\omega_n} = 1 \quad \phi = \pi/2$
$\zeta = \text{any value} \quad \frac{\omega}{\omega_n} \geq 1 \quad \phi = \pi$

from the vectorial representation

$$f_0 = \sqrt{(Kx - m\omega^2 x)^2 + (c\omega x)^2}$$

$$\frac{f_0}{K} = X_{st} = X\sqrt{\left(\frac{K}{K} - \frac{m\omega^2}{K}\right)^2 + \left(\frac{c\omega}{K}\right)^2}$$

Where X_{st} = zero frequency deflection of the system

and $X_{st} = X\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + 2\zeta\left(\frac{\omega}{\omega_n}\right)^2}$

and $\dfrac{X_{st}}{X} = \sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + 2\zeta\left(\frac{\omega}{\omega_n}\right)^2}$

where $\dfrac{X}{X_{st}}$ = Magnification factor.

$$\Rightarrow \boxed{\frac{X}{X_{st}} = \frac{1}{\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}} \quad \text{— (6)}$$

At resonance $\boxed{\omega = \omega_n, \quad \frac{X}{X_{st}} = \frac{1}{2\zeta}}$ = Magnification factor. — (7)

so the amplitude of vibration

$$X = \frac{X_{st}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}} \quad \text{—(8)}$$

and phase lag $\phi = \tan^{-1}\left[\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]$

We have the particular solution

$$x_p = X\sin(\omega t - \phi) \quad \text{—(9)}$$

substituting the value of X in eq. (9) we have

$$\boxed{x_p = \frac{X_{st}\sin(\omega t - \phi)}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left\{2\zeta\left(\frac{\omega}{\omega_n}\right)\right\}^2}}} \quad \text{—(10)}$$

In forced vibration $\left(\frac{\omega_p}{\omega_n}\right) = \sqrt{1-2\zeta^2}$ —(11)

Also $\frac{\omega_d}{\omega_n} = \sqrt{1-\zeta^2}$ —(12)

where ω_p = frequency corresponding to the peak amplitude.

Example-1

A damped natural frequency of a system as obtained from a free vibration test is 9.8 Hz. During the forced vibration test with constant excitation force on the same system max^m amplitude of vibration is found to be 9.6 Hz. Find the damping factor for the system and it's natural frequency.

Given! $\omega_p = 9.6\ Hz$ $\qquad \omega_d = (9.8 \times 2\pi)$ rad/s.

$= (9.6\times 2\pi)$ rad/sec.

we have the relation $\frac{\omega_p}{\omega_n} = \sqrt{1-2\zeta^2}$

$\Rightarrow \frac{9.6\times 2\pi}{\omega_n} = \sqrt{1-2\zeta^2}$

also $\frac{\omega_d}{\omega_n} = \sqrt{1-\zeta^2}$

$\Rightarrow \frac{9.8\times 2\pi}{\omega_n} = \sqrt{1-\zeta^2}$

Dividing the two equations

$$\frac{9.6}{9.8} = \frac{\sqrt{1-2\zeta^2}}{\sqrt{1-\zeta^2}}$$

$$\Rightarrow \boxed{\zeta = 0.196}$$

Substituting the value of $\zeta = 0.196$ in any of the two equations

$$\omega_n = (10 \times 2\pi) \text{ rad/s.}$$

$$\text{or } f_n = \frac{\omega_n}{2\pi} = 10 \text{ Hz} \quad (\text{Ans})$$

<u>Example - 2</u>

Consider a spring-mass-damper system with $k = 4000$ N/m, $m = 10$ kg and $c = 40$ N-s/m. Find the steady state and total response of the system under the harmonic force $F = 200 \sin 10t$ N for initial conditions $x = 0.1$ m and $\dot{x} = 0$, at $t = 0$.

Given:- $k = 4000$ N/m, $m = 10$ kg, $c = 40$ N-s/m

$$\text{So } \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{10}} = 20 \text{ rad/s.}$$

$$\text{Now } \zeta = \frac{c}{c_c} = \frac{c}{2m\omega_n} = \frac{40}{2 \times 10 \times 20} = 0.1$$

$$\omega_d = \sqrt{1-\zeta^2} \cdot \omega_n = \sqrt{1-0.1^2} \times 20 = 19.9 \text{ rad/s.}$$

Steady state amplitude

$$X = \frac{X_{st}}{\sqrt{\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left\{2\zeta\left(\frac{\omega}{\omega_n}\right)\right\}^2}}$$

$$\Rightarrow X = \frac{f_0/k}{\sqrt{\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left\{2\zeta\frac{\omega}{\omega_n}\right\}^2}}$$

$$= \frac{200/4000}{\sqrt{\left\{1-\left(\frac{10}{20}\right)^2\right\}^2 + \left(\frac{2 \times 0.1 \times 10}{20}\right)^2}}$$

$$= 0.066 \text{ m.}$$

$$\text{phase lag } \phi = \tan^{-1}\left[\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right] = \tan^{-1}\left[\frac{2 \times 0.1 \times (10/20)}{\left\{1-\left(\frac{10}{20}\right)^2\right\}}\right]$$

$$= 7.59^\circ$$

The steady state response of the system is given by

$x_p = X \sin(\omega t - \phi)$

$= 0.066 \sin(10t - 7.59^\circ)$

The transient response $x_c = Ae^{-\zeta\omega_n t} \sin(\omega_d t + \phi_1)$

Total response of the system

$x = x_c + x_p$

$= Ae^{-\zeta\omega_n t} \sin(\omega_d t + \phi_1) + 0.066 \sin(10t - 7.59^\circ)$ —(1)

The values of A and ϕ_1 are calculated from the initial conditions.

Now differentiating eq. (1) we have

$\dot{x} = -\zeta\omega_n Ae^{-\zeta\omega_n t} \sin(\omega_d t + \phi_1) + A\omega_d e^{-\zeta\omega_n t} \cos(\omega_d t + \phi_1)$

$+ .066 \times 10 \cos(10t - 7.59^\circ)$ —(2)

substituting the initial conditions

$0.1 = A \sin\phi_1 + 0.066 \sin(-7.59^\circ)$

$\Rightarrow 0.1 = A \sin\phi_1 - 0.0087$

$\Rightarrow \boxed{A \sin\phi_1 = 0.1087}$ —(3)

from eq. (2) $0 = -\zeta\omega_n A \sin\phi_1 + A\omega_d \cos\phi_1 + 0.654$

$\Rightarrow \boxed{A \cos\phi_1 = -0.020}$ —(4)

$\tan\phi_1 = \dfrac{0.1087}{-0.020} \Rightarrow \boxed{\phi_1 = -79.57^\circ}$

and $\boxed{A = 0.11}$

So the total response of the system is given by

$x = 0.11 e^{-2t} \sin(19.9t - 79.57^\circ) + 0.066 \sin(10t - 7.59^\circ)$ (Ans).

Example 3

Find the natural ~~frequency~~ response of a single dof system with $m = 10$ kg, $c = 50$ N-s/m, $k = 2000$ N/m under the action of harmonic force $F = F_0 \sin\omega t$ with $F_0 = 200$ N and $\omega = 31.416$ rad/s. The initial conditions may be assumed as $x = 0.01$ m and $\dot{x} = 5 m/s$ at $t = 0$.

from the given data $\omega_n = \sqrt{\frac{K}{m}} = 14.142$ rad/s.

$$\xi = \frac{c}{c_c} = \frac{c}{2m\omega_n} = \frac{50}{2 \times 10 \times 14.142} = 0.1768$$

$$\omega_d = \sqrt{1-\xi^2}\,\omega_n = 13.92 \text{ rad/s.}$$

$$X_{st} = \frac{F_0}{K} = \frac{200}{2000} = 0.1 \text{ m.}$$

steady state amplitude

$$X = \frac{X_{st}}{\sqrt{\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left\{2\xi\frac{\omega}{\omega_n}\right\}^2}} = \frac{0.1}{\sqrt{\left\{1-\left(\frac{31.416}{14.142}\right)^2\right\}^2 + \left(\frac{2 \times .1768 \times 31.416}{14.142}\right)^2}}$$

$$\Rightarrow \boxed{X = 0.0249 \text{ m.}}$$

$$\phi = \tan^{-1}\left[\frac{2\xi\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]$$

$$= -11.53^\circ$$

Total response of the system is given by

$$x = x_c + x_p$$

$$= Ae^{-\xi\omega_n t}\sin(\omega_d t + \phi_1) + X\sin(\omega t - \phi)$$

$$= A e^{-2.5t}\sin(13.92t + \phi_1) + 0.0249\sin(31.416t + 11.53^\circ) \quad \text{— (1)}$$

Differentiating eq. (1) wrt time

$$\dot{x} = -2.5Ae^{-2.5t}\sin(13.92t + \phi_1) + 13.92Ae^{-2.5t}\cos(13.92t + \phi_1) + (0.0249)(31.416)\cos(31.416t + 11.53^\circ) \quad \text{— (2)}$$

Applying initial conditions we have

$$0.01 = A\sin\phi_1 + 0.0249\sin 11.53^\circ$$

$$0.01 = A\sin\phi_1 + 0.0049$$

$$\Rightarrow \boxed{A\sin\phi_1 = 0.005} \quad \text{— (3)}$$

and $5 = -2.5A\sin\phi_1 + 13.92A\cos\phi_1 + 0.766$

$$\Rightarrow \boxed{A\cos\phi_1 = 0.305} \quad \text{— (4)}$$

$$\boxed{\phi_1 = 0.94^\circ}$$

$$A = \cancel{\tan^{-1}\left(\frac{0.005}{0.305}\right)} = \boxed{0.3}$$

The total response

$$x = 0.3\,e^{-2.5t}\sin(13.92t + 6.94^\circ) + 0.0249\sin(31.416t + 11.53^\circ)$$

Example-4

find out the frequency ratio for which amplitude in forced vibration will be maximum. Also determine the peak amplitude and the corresponding phase angle.

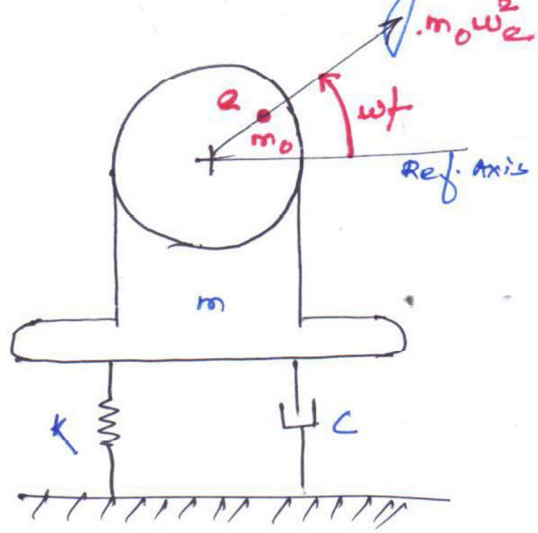
Forced Vibration with Rotating and Reciprocating Unbalance:

All rotating machinery like electric motor, turbine etc. have some amount of unbalance left in them after correcting their unbalance on precession balancing m/c.

Let m_0 = an equivalent mass rotating with its centre of gravity 'e' from axis of rotation.

- Then the final unbalance is measured in terms of the equivalent mass m_0 rotating with its centre of gravity at a distance 'e' from the axis of rotation.

The centrifugal force generated because of the rotation of the body is proportional to the square of the frequency of rotation. This force is max^m value of the sinusoidal excitation in any direction.



Consider an elastically supported m/c rotating at 'ω' rad/s. Let the unbalance mass m_0 have an eccentricity 'e'.

Let m = total mass of the m/c including m_0

k = spring stiffness

c = damping coefficient

Let m_0 mass makes an angle ωt with the reference axis at any instant.

The equation of motion in vertical axis is:

$$(m-m_0)\frac{d^2x}{dt^2} + m_0\frac{d^2}{dt^2}\left(x+e\sin\omega t\right) = -Kx - c\dot{x}$$

or $$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + Kx = m_0 e\omega^2 \sin\omega t \quad \text{---}(1)$$

Comparing Eq.(1) with that of the eq. of motion for a forced vibration of single dof system

f_0 is replaced by $m_0 e \omega^2$

Therefore the steady state amplitude is given by

$$\boxed{X = \frac{m_0 e \omega^2 / k}{\sqrt{\left(1 - \frac{m\omega^2}{k}\right)^2 + \left(\frac{c\omega}{k}\right)^2}}} \qquad \text{— (2)}$$

In a dimensionless form

$$\boxed{\frac{X}{\left(\frac{m_0 e}{m}\right)} = \frac{(\omega/\omega_n)^2}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}} \qquad \text{— (3)}$$

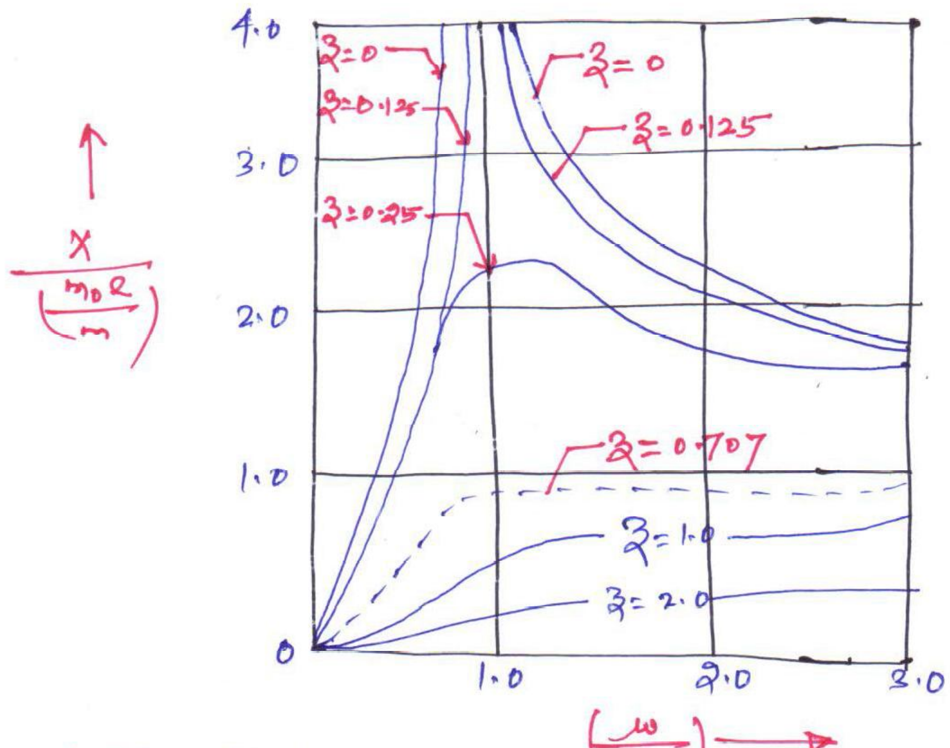
Phase lag

$$\boxed{\phi = \tan^{-1}\left[\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right]} \qquad \text{— (4)}$$

At low speed the centrifugal exciting force $m_0 e \omega^2$ is small and therefore the response curve starts from zero.

At resonance $\omega/\omega_n = 1$ and

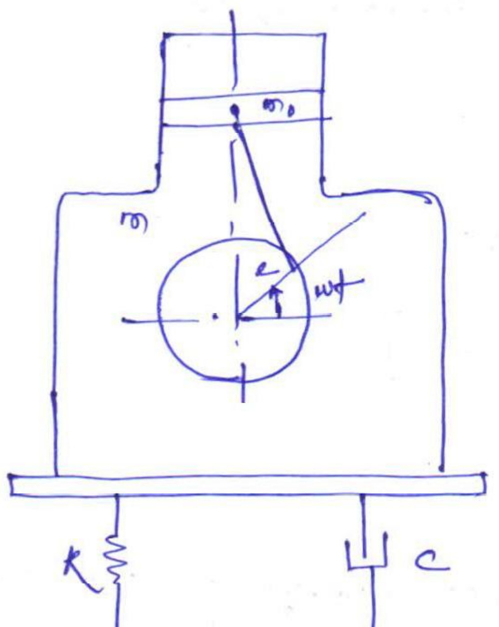
$$\boxed{\frac{X}{\left(\frac{m_0 e}{m}\right)} = \frac{1}{2\zeta}} \qquad \text{— (5)}$$



(Dimensionless amplitude $\left(\frac{\omega}{\omega_n}\right)$ frequency ratio plot)

Vibration analysis of Reciprocating mass:-

Let m_0 = equivalent mass of reciprocating part
m = total mass of the engine including the reciprocating mass.
e = crank length
l = length of connecting rod.



The inertia force due to the reciprocating mass is approximately
$= m_0 e\omega^2 \left\{ \sin\omega t + \left(\frac{e}{l}\right) \sin 2\omega t \right\}$
If 'e' is small compared to l, the second harmonic may be neglected and the exciting force becomes equal to $m_0 e\omega^2 \sin\omega t$ and is same as in case of rotating unbalance mass.
Therefore for small 'e' same vibration analysis is followed in case of reciprocating unbalance mass.

Example-1

A system of beam supports a motor of mass 1200 kg. The motor has an unbalanced mass of 1 kg located at 6 cm radius. It is known that resonance occurs at 2210 rpm. What amplitude of vibration can be expected at motor's operating speed of 1440 rpm if damping factor is 0.1 and 0 respectively.

We have $\frac{\omega}{\omega_n} = \frac{1440}{2210} = 0.652$

$\frac{m_0}{m} = \frac{1}{1200}$, $e = 0.06\ m$

$\xi = 0.1$

Using the relation

$$\frac{x}{\left(\frac{.06}{1200}\right)} = \frac{(0.652)^2}{\sqrt{\{1 - 0.652^2\}^2 + \{2 \times 0.1 \times 0.652\}^2}}$$

$\Rightarrow \boxed{x = 0.036\ mm}$

If $\xi = 0$

$$\frac{x}{\left(\frac{.06}{1200}\right)} = \frac{(0.652)^2}{[1 - 0.652^2]}$$

$\Rightarrow \boxed{x = 0.037\ mm}$

Example-2

A single cylinder vertical petrol engine of total mass 320 kg is mounted upon a steel chassis frame and causes a vertical static deflection of 0.2 cm. The reciprocating parts of the engine have a mass of 24 kg. and move through a vertical stroke of 15cm with SHM. A dashpot is provided, the damping resistance of which is directly proportional to the velocity and amounts to 490 N at 0.3 m/s. Determine

(a) the speed of driving shaft at which resonance will occur

(b) amplitude of steady state forced vibration when the driving shaft of the engine rotates at 480 rpm.

Let $m = 320$ Kg $\quad \Delta st = 0.002$ m $\quad m_0 = 24$ Kg.

$$e = \frac{0.15}{2} = 0.075 \text{ m}.$$

$$\omega_n = \sqrt{\frac{g}{\Delta st}} = \sqrt{\frac{9.81}{0.002}} = 70 \text{ rad/s}.$$

resonance speed $= \frac{70}{2\pi} \times 60 = 670$ rpm

$$\omega = \frac{480 \times 2\pi}{60} = 50.4 \text{ rad/sec}.$$

$$\text{so } \left(\frac{\omega}{\omega_n}\right) = \frac{50.4}{70} = 0.72$$

$$\zeta = \frac{c}{2m\omega_n} = \frac{490/0.3}{2 \times 320 \times 70} = 0.0364$$

$$\frac{m_0}{m} = \frac{24}{320} = 0.075$$

$$\text{Now } \frac{X}{\left(\frac{m_0 e}{m}\right)} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

$$\Rightarrow \frac{X}{0.075 \times .075} = \frac{(0.72)^2}{\sqrt{(1 - .72)^2 + (2 \times .0364 \times .72)^2}}$$

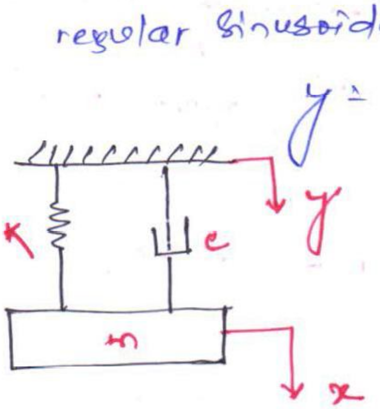
$$\Rightarrow \boxed{X = 0.006 \text{ m or } 6 \text{ mm}}$$

forced vibration due to base excitation:-

In most of the vibration related problems, a system is being excited by motion of the support, for example a vehicle is travelling on a wavy road, an engine mounted on a vibrating system etc.

- In this case the support is considered to be excited by a regular sinusoidal motion,

$$y = Y \sin \omega t \quad \text{---} (1)$$



Considering a spring-mass damper system the mass is attached with the support by means of a spring of stiffness K, a damper of damping coefficient c.