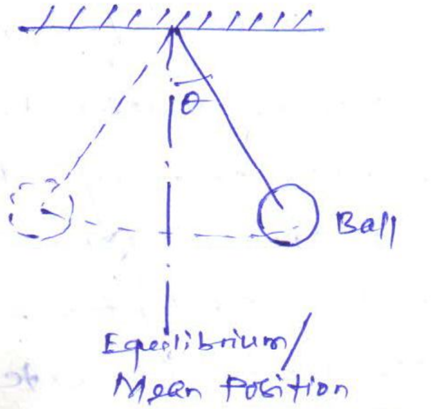
MODULE 1

SINGLE DEGREE OF FREEDOM VIBRATION SYSTEM

MODULE-1

Basic Concept of Vibration / What is Vibration?

When body particles are displaced by the application of external force, the internal forces in the form of elastic energy are present in the body. These forces try to bring the body to its original position. At equilibrium position, the entire elastic energy is converted into kinetic energy and the body continues to move in the opposite direction and the process repeats.



(Fig-1 A simple pendulum)

— So any motion which repeats itself after an interval of time is called vibration; e.g. simple pendulum (shown in fig-1)

Reasons of Vibrations:-

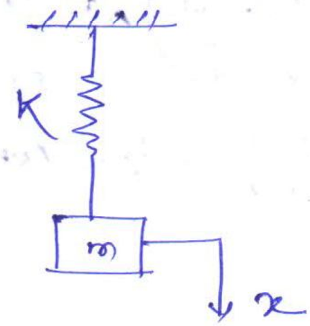
1. Unbalanced forces in the machine :- forces produced within the machine
2. Dry friction between two mating surfaces :- This produces a self excited vibration
3. External excitations :- The excitations may be periodic, random etc.
4. Earthquake : Responsible for failure of buildings/ dams etc.
5. Wind :- It may cause vibration of transmission and telephone lines under certain condition.

Definitions:-

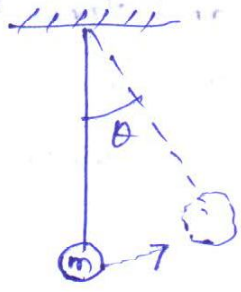
1. Periodic motion → A motion which repeats itself after equal interval of time.
2. Time period → Time taken to complete one cycle.
3. Frequency → No of cycles/unit time
4. Simple Harmonic Motion → A periodic motion of a particle whose acceleration is always directed towards the mean position.
5. Amplitude of motion → Maximum displacement of a vibrating body from mean position
6. Free vibrations → Vibration of a system because of its own elastic property without any external exciting forces acting on it.
7. Forced vibration → The vibrations the system executes under the action of an external periodic force. The frequency of vibration is same to that of excitation.
8. Natural frequency → frequency of free vibration of the system. It is constant for a given system.
9. Resonance : → Vibration of a system ~~when~~ in which the frequency of external force is equal to the natural frequency of the system.
10. Damping → Resistance to the motion of the vibrating body.
11. Degree of freedom → No of independent coordinates required to specify completely the configuration of the system at any instant.

Few examples of Single degree of freedom system, have been

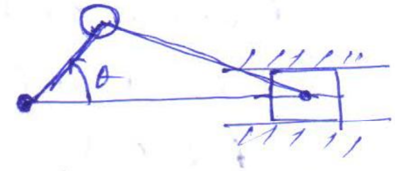
Example :-



(a) Spring mass system



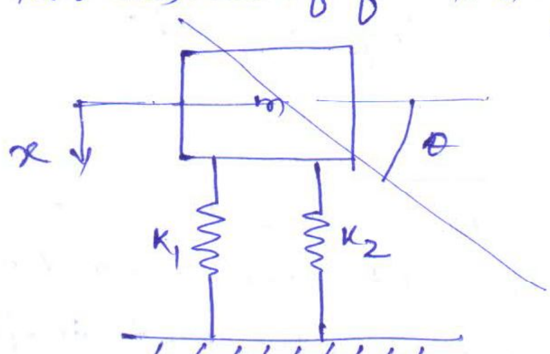
(b) Simple pendulum



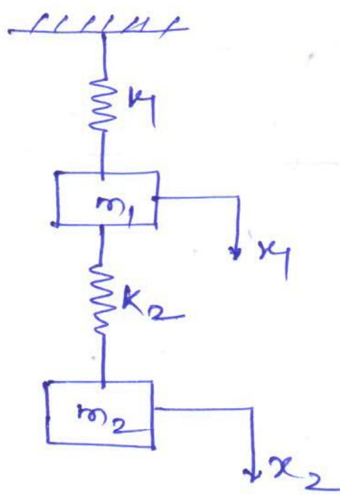
(c) Crank slider Mechanism.

(Fig 2 :- Examples of Single Degrees of Freedom system.)

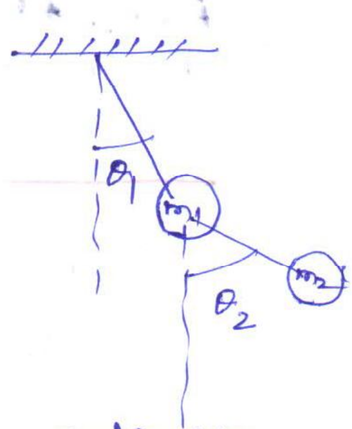
shown in Fig. (2). And Fig (3) depicts few examples of two degrees of freedom system.



(a) Spring supported Rigid mass.

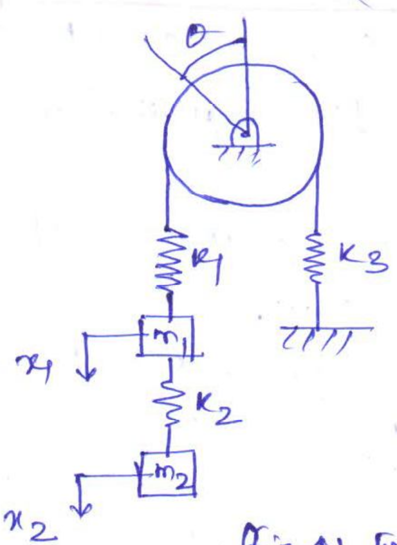


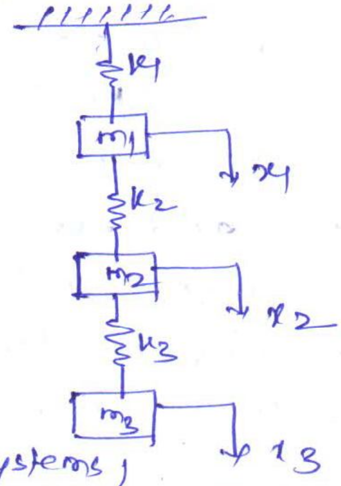
(b) Two mass two spring system.



(c) Double pendulum system.

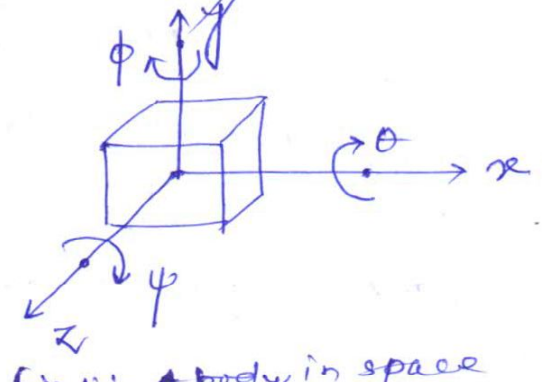
[Fig - 3 :- Examples of two degrees of freedom (DOF) systems]



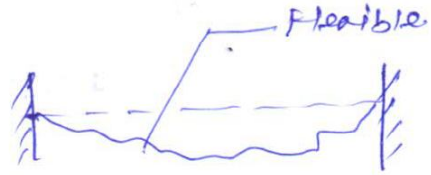


(Fig 4: Example of 3 DOF systems)

Similarly a rigid body in space has six dof (i.e. three translational and three rotational), as shown fig. 5. And a flexible beam beth two supports has a infinite no of degrees of freedom (shown in fig. 6).

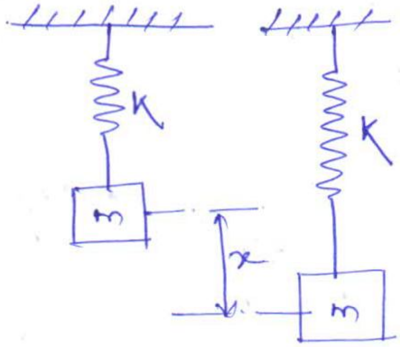


(Fig 5: A body in space having 6 dof)



(Fig. 6: A flexible beam having infinite dofs)

Derivation of Differential Equation:-



(Fig- 7: spring mass system)

Consider a spring mass system (Fig 7) constrained to move in a rectilinear manner along the longitudinal axis.

Let m = mass of the block attached to spring

K = Spring stiffness.

$$\boxed{\begin{array}{l}\text{Sign convection} \to \\ \text{Downward} = +ve \\ \text{Upword} = -ve\end{array}}$$

- At any instant let the mass occupy any displaced position,

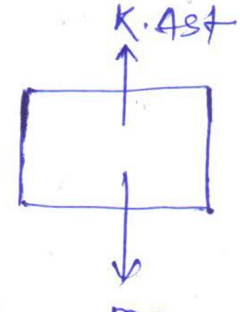
Let x = displacement of mass m from equilibrium position.

Considering displacement x to be +ve in downward direction and -ve in the upward direction.

For an initial infinitesimal displacement of Δst, prior to x displacement, in the equillibrium position the forces acting on the mass are:

(i) mg → vertically downward

(ii) $K \cdot \Delta st$ → spring force, vertically upward.



(Fig 8: Eq. position of mass)

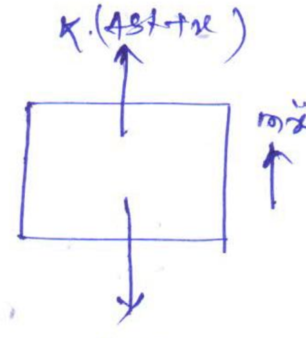
For equilibrium

$$mg = K \cdot \Delta st \quad \text{--- (1)}$$

And after a displacement of 'x'

total spring force $= K \cdot (\Delta st + x)$

And the forces acting on the mass from Newton's 2nd law of motion



(Fig. 9: forces after x displacement)

$$m\ddot{x} = mg - K \cdot (\Delta st + x)$$
$$= mg - K \cdot \Delta st - Kx$$
$$= \cancel{mg} - \cancel{mg} - Kx \quad \{\text{from eq. (1)}\}$$
$$\Rightarrow \boxed{m\ddot{x} + Kx = 0} \quad \text{--- (2)}$$

<u>Solution of Differential Equation:-</u>

We have the differential equation for the spring mass system

$$m\ddot{x} + Kx = 0$$

It is an equation of simple harmonic motion.

~~The solution of the above equation will be~~

$$\cancel{x = A\cos\omega_n t + B\sin\omega_n t \quad \text{--- (2)}}$$

Now from eq. (1) we have

$$\ddot{x} + \frac{K}{m}x = 0 \quad \text{--- (3)}$$

let $\frac{K}{m} = \omega_n^2$

so equation (3) may be written as:

$$\boxed{\ddot{x} + \omega_n^2 x = 0} \quad \text{--- (4)}$$

~~Eq. (4) has a solution as in eq. (2)~~

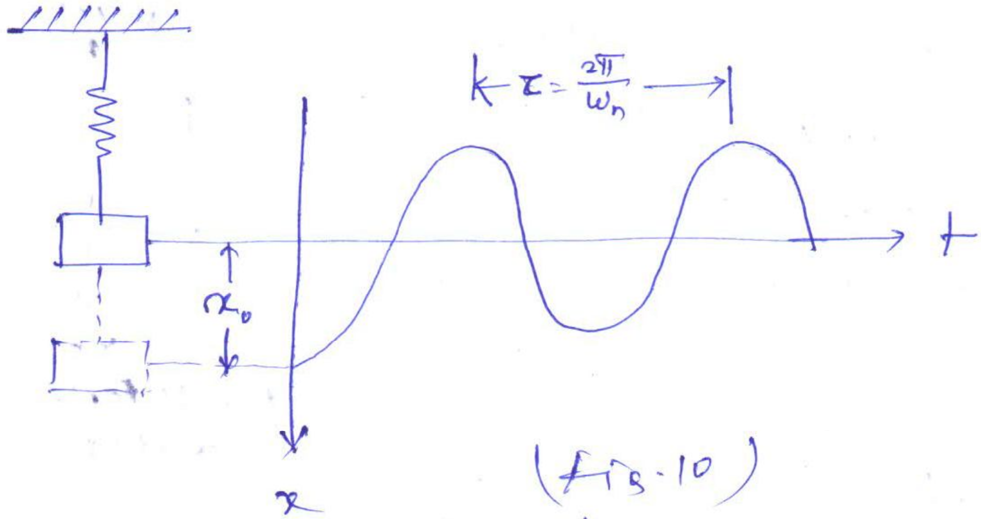
$$\cancel{x = A\cos\omega_n t + B\sin\omega_n t}$$

The standard solution for this differential equation is written as

$$x = A \sin \omega_n t + B \cos \omega_n t \quad \text{--- (5)}$$

Where A and B are constants, whose value can be obtained from initial conditions.

$$\left.\begin{aligned} x &= x_0 , \quad \text{at } t = 0 \\ \dot{x} &= 0 , \quad \text{at } t = 0 \end{aligned}\right\} \text{--- (6)}$$



(fig. 10)

Differentiating equation (5)

$$\dot{x} = A\omega_n \cos \omega_n t - B \omega_n \sin \omega_n t \quad \text{--- (7)}$$

substituting the initial condition in eq. (5) and eq. (7)

$$X_0 = 0 + B$$

$$0 = A\omega_n - 0$$

Gives $\boxed{\begin{aligned} A &= 0 \\ B &= X_0 \end{aligned}}$ --- (8)

Substituting the values of these constants, we have

$$\boxed{x = X_0 \cos \omega_n t} \quad \text{--- (9)}$$

Equation (9) is the final solution for the specified initial condition.

The time period for one complete cycle of 2π rad is

$$\tau = \frac{2\pi}{\omega_n}$$

Natural frequency is the inverse of time period

$$f_n = \frac{\omega_n}{2\pi}$$

$$\Rightarrow \boxed{\omega_n = \sqrt{\frac{K}{m}}}$$

Therefore $f_n = \frac{1}{2\pi}\sqrt{\frac{K}{m}} = \frac{1}{2\pi}\sqrt{\frac{K \cdot g}{m \cdot g}} = \frac{1}{2\pi}\sqrt{\frac{g}{\Delta st}}$ $\left(\because \Delta st = \frac{mg}{K}\right)$

or $\boxed{f_n = \frac{1}{2\pi}\sqrt{\frac{9.8}{\Delta st}} = \frac{0.4892}{\sqrt{\Delta st}}}$ Hz. — (10)

Example-1 A light cantilever of length l has a mass M fixed at it's free end. Find the frequency of lateral vibration in the vertical plane.

The deflection at the free end of the cantilever

$$\Delta st = \frac{M \cdot g l^3}{3EI} \quad \text{— (1)}$$

Where E = modulus of elasticity
I = MI of the section of beam about it's neutral axis.

Now stiffness $K = \frac{Mg}{\Delta st} = Mg \times \frac{3EI}{Mgl^3} = \boxed{\frac{3EI}{l^3}}$

and circular frequency $\omega_n = \sqrt{\frac{K}{m}}$

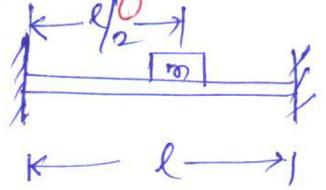
$$= \sqrt{\frac{3EI/l^3}{M}} = \sqrt{\frac{g}{\Delta st}}$$

$\Rightarrow$ ~~$\omega_n = \sqrt{\frac{g}{\Delta st}}$ rad/sec~~

$$\Rightarrow \boxed{\omega_n = \sqrt{\frac{3EI}{l^3 M}}}$$

And natural frequency $\boxed{f_n = \frac{1}{2\pi}\sqrt{\frac{3EI}{Ml^3}}}$ Hz.

Example-2 Find the natural frequency of the sys. shown in the figure.



Deflection at the centre of a beam fixed at both ends and a central load W is

$$\Delta st = \frac{Wl^3}{192EI}$$

and stiffness $K = \dfrac{\text{load}}{\text{deflection}} = \dfrac{W}{Wl^3/192EI}$

$$\Rightarrow \boxed{K = \frac{192EI}{l^3}}$$

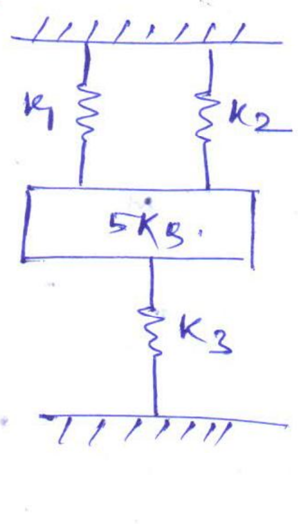
General equation for undamped free vibration is

$$m\ddot{x} + Kx = 0$$

and $\omega_n = \sqrt{\dfrac{K}{m}} = \sqrt{\dfrac{192EI}{ml^3}}$ rad/sec.

so Natural frequency $f_n = \dfrac{1}{2\pi}\sqrt{\dfrac{192EI}{ml^3}}$ Hz.

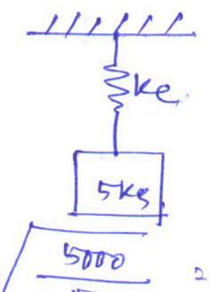
Example-3 Find the natural frequency of the system shown in figure. Given $k_1 = k_2 = 1500$ N/m, $K_3 = 2000$ N/m, $m = 5$ kg.



The equivalent stiffness for springs in parallel

$K_e = K_1 + K_2 + K_3 = 1500 + 1500 + 2000$

$= 5000$ N/m.

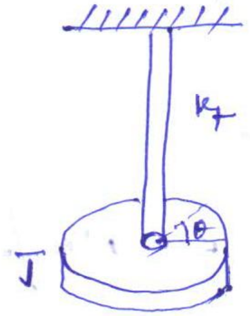


so $\omega_n = \sqrt{\dfrac{K_e}{m}} = \sqrt{\dfrac{5000}{5}} = 31.62$ rad/sec.

$f_n = \dfrac{1}{2\pi}\sqrt{1000} = 5.03$ Hz. (Ans).

Torsional Vibrations:-

Considering a rotor ~~mass~~ having a mass moment of inertia J, connected to a shaft of torsional stiffness K_t (as shown in fig-11): When the rotor is displaced in an angular manner, it executes torsional vibrations.
- Its natural frequency can be obtained in the following manner:



(fig-11 Torsional system)

At any instance the rotor occupies a position θ with reference to the equilibrium position. The torque acting on the rotor $= -K_t\theta$

-ve sign indicates the torque acts on the rotor ~~rotor~~ in opposite direction to that of the twist.

$$J\ddot{\theta} = -K_t\theta \quad \text{--- (1)}$$

or $J\ddot{\theta} + K_t \cdot \theta = 0$

or $\ddot{\theta} + \left(\frac{K_t}{J}\right)\theta = 0 \quad \text{--- (2)}$

substituting $\omega_n^2 = K_t / J \quad \text{--- (3)}$

So equation (2) becomes

$$\boxed{\ddot{\theta} + \omega_n^2\theta = 0} \quad \text{--- (4)}$$

Natural frequency of vibration of this system can be obtained from the equation

$$\boxed{\omega_n = \sqrt{K_t/J}} \quad \text{--- (5)}$$

so $$\boxed{f_n = \frac{1}{2\pi}\sqrt{\frac{K_t}{J}}}$$

Example-4 Calculate the natural frequency of vibration of a torsional pendulum with following dimensions:

length of the rod, $l = 1$ m.
Diameter of rod, $d = 5$ mm
Diameter of rotor, $D = 0.2$ m
Mass of rotor, $M = 2$ kg.

The modulus of rigidity for the material of rod may be assumed to be $0.83 \times 10^{11}\ N/m^2$

Soln We have mass moment of inertia $J = \frac{1}{2} mr^2$

$\Rightarrow J = \frac{1}{2} M \left(\frac{D}{2}\right)^2 = \frac{1}{2} \times 2 \times (0.1)^2 = 0.01\ kg\,m^2$

Now using the relation

$$\frac{T}{I_p} = \frac{G\theta}{l}$$

or, torsional stiffness $k_t = \frac{T}{\theta} = \frac{G \cdot I_p}{l}$

so $k_t = \dfrac{0.83 \times 10^{11} \times \frac{\pi}{32} \times (0.005)^4}{1} = 5.09\ Nm/rad$

so $\omega_n = \sqrt{\frac{k_t}{J}} = \sqrt{\frac{5.09}{0.01}} = 22.6\ rad/sec$

$f_n = 22.6/2\pi = \boxed{3.59\ Hz}$

Energy Method:-

free vibration of systems involves the cyclic interchange of KE and PE. In undamped free vibrating systems no energy is dissipated or removed from the system. The KE, T is stored in the mass by virtue of it's velocity and potential energy U is stored in the form of strain energy in elastic deformation. As the total energy in the system is constant the principal of conservation of mechanical energy applies. Since the mechanical energy is conserved, the sum of KE and PE is constant and it's rate of change of is zero.

The principle can be expressed as

$$\boxed{\begin{gathered} T + U = \text{constant} \\ \frac{d}{dt}(T+U) = 0 \end{gathered}}$$

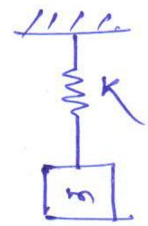
Energy Method :-

free vibration of systems involves the cyclic interchange of KE and PE. In undamped free vibrating systems, no energy is dissipated or removed from the system. The KE ~~is stored~~ T is stored in the mass by virtue of it's velocity and potential energy U is stored in the form of strain energy in elastic deformation. As the total energy in the system is constant, the principle of conservation of mechanical energy applies. Since the mechanical energy is conserved, the sum of KE and PE is const and it's rate of change is zero.

The principle can be expressed as:

$$T + U = \text{constant}$$

$$\frac{d}{dt}(T+U) = 0$$



For the system shown in the figure

$$T = \frac{1}{2} m\dot{x}^2$$

$$U = \frac{1}{2} Kx^2$$

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}Kx^2\right) = 0$$

$$\Rightarrow (m\ddot{x}\dot{x} + K\dot{x}x) = 0$$

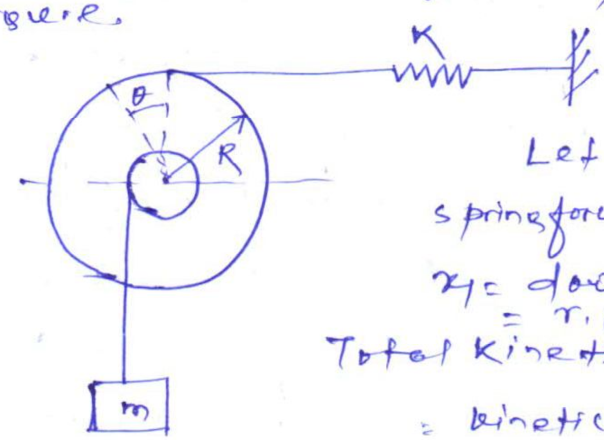
$$\Rightarrow \boxed{m\ddot{x} + Kx = 0}$$

And $\omega_n = \sqrt{\frac{K}{m}}$

natural frequency $\boxed{f_n = \frac{1}{2\pi}\sqrt{\frac{K}{m}}}$

Example - 1

Find the natural frequency of the system shown in the figure.



Let x_2 = spring deflection

spring force $K \cdot x_2 = KR \cdot \theta$

x_1 = downward movement of mass m $= r \cdot \theta$

Total Kinetic energy

= kinetic energy of mass + kinetic energy of rotating element

$= \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} I \dot{\theta}^2$

Potential energy of spring $= \frac{1}{2} K x_2^2$

Total energy $= \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} K x_2^2$

By energy method, we have

$$\frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} K x_2^2 = \text{constant} \quad \text{--- (1)}$$

~~Differenciating eq.(1) wrt time~~

It may be represented as

$$\frac{1}{2} m r^2 \dot{\theta}^2 + \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} K R^2 \theta^2 = \text{constant}$$

$$\frac{d}{d\theta}\left[\frac{1}{2} m r^2 \dot{\theta}^2 + \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} K R^2 \theta^2\right] = \frac{d}{d\theta}(\text{constant})$$

$$m r^2 \dot{\theta} \ddot{\theta} + I \dot{\theta} \ddot{\theta} + K R^2 \theta \dot{\theta} = 0$$

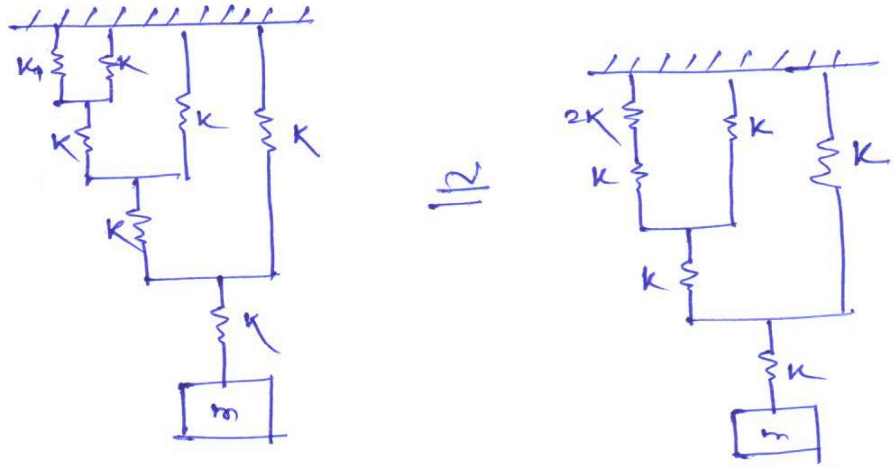
$$(m r^2 + I)\ddot{\theta} + K R^2 \theta = 0$$

$$\text{or} \quad \ddot{\theta} + \left(\frac{K R^2}{m r^2 + I}\right)\theta = 0$$

$$\omega_n = \sqrt{\left(\frac{K R^2}{m r^2 + I}\right)} \quad \text{or} \quad f_n = \frac{1}{2\pi}\sqrt{\frac{K R^2}{m r^2 + I}}$$

(Ans)

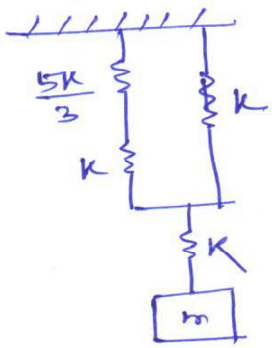
Example - 2 find the natural frequency of the system shown in the figure. Take $K = 2 \times 10^5$ N/m, $m = 20$ kg.



$$\frac{1}{k_{e1}} = \frac{1}{2k} + \frac{1}{k} \Rightarrow \frac{1}{k_{e1}} = \cancel{\left(\frac{1+2k}{2k}\right)} \Rightarrow k_{e1} = \frac{2K}{3}$$

$$\cancel{k_{e2} = \frac{2k}{1+2k} + k = \frac{2k+k+2k^2}{1+2k}}$$

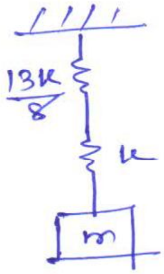
$$k_{e2} = \frac{2K}{3} + K = \frac{5K}{3}$$



$$\frac{1}{k_{e3}} = \frac{3}{5K} + \frac{1}{K} = \frac{3+5}{5K}$$

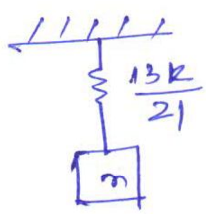
$$k_{e3} = \frac{5K}{8}$$

$$k_{e4} = \frac{5K}{8} + K = \frac{13K}{8}$$



$$\frac{1}{k_{e5}} = \frac{8}{13K} + \frac{1}{K} = \frac{8+13}{13K}$$

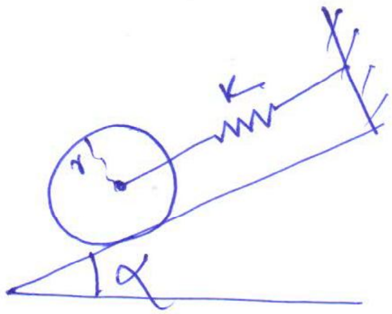
$$k_{e5} = \frac{13K}{21}$$



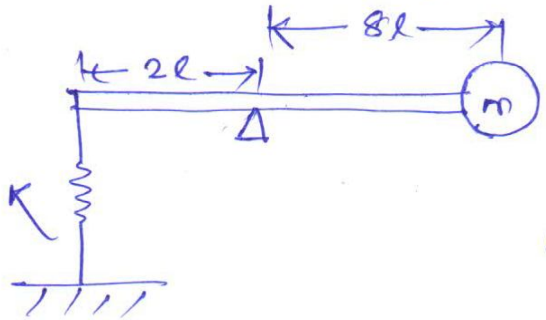
$$\omega_n = \sqrt{\frac{k_e}{m}} = \sqrt{\frac{13 \times 2 \times 10^5}{20 \times 21}} = 78.68 \text{ rad/sec}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k_e}{m}} = 12.5 \text{ Hz.}$$

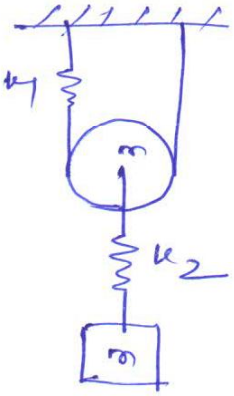
Assignment - 1

① 

A circular cylinder of radius r and mass m is connected by a spring of stiffness K on an inclined plane. If it is free to roll on the rough surface which is without slipping, determine the natural frequency.

② 

Find the natural frequency of the system if $m = 10\,kg$, $K = 1000\,N/m$

③ 

Determine the natural frequency of the mass $m = 15\,kg$,

$k_1 = 8 \times 10^3\,N/m$

$k_2 = 6 \times 10^3\,N/m$,