MODULE 2

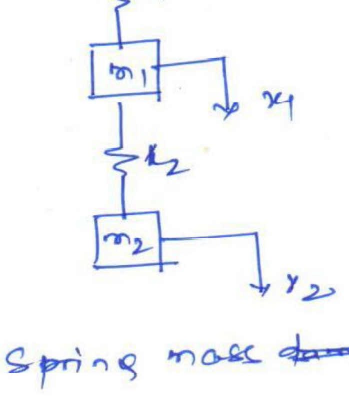
MODULE - II

— : TWO DEGREE OF FREEDOM SYSTEMS : —

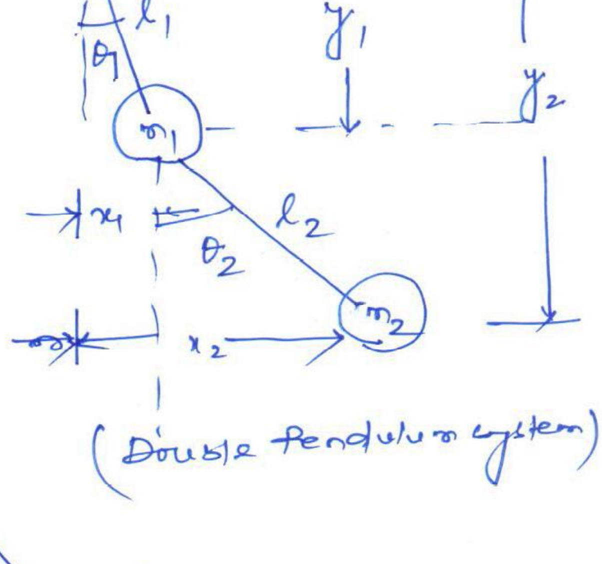
In the preceding sections, systems having single dof have been discussed. In this case the systems have one natural frequency and require only one independent coordinate ~~to~~ describe the system completely. Systems having two dof are important and they introduce the coupling phenomenon where the motion of any of the two independent coordinates, depend also on the other coordinate through spring coupling or dashpots.

- These systems require two independent coordinates to describe their motion.

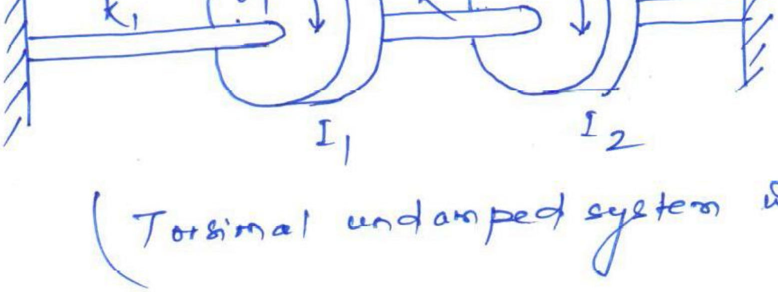
Examples :—



Spring mass ~~damp~~ system

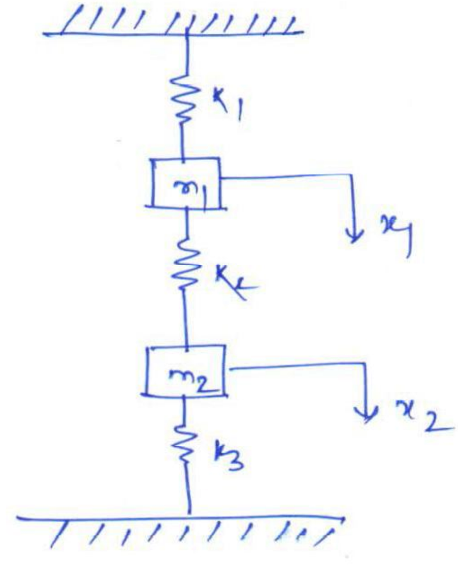


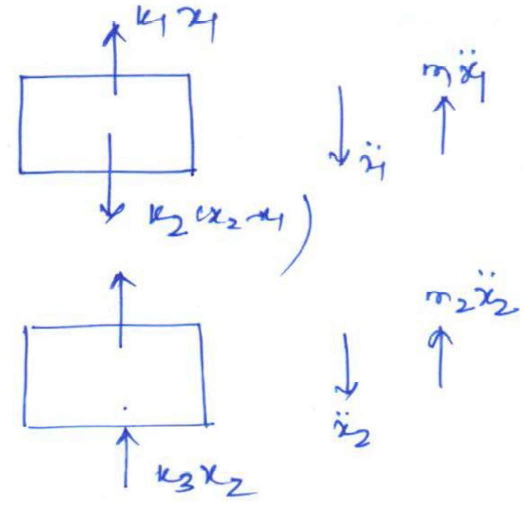
(Double pendulum system)



(Torsional undamped system with two mass)

Principal mode of vibration :-





considering an ideal case of two dof system (spring mass system)

Let x_1, x_2 $\rightarrow$ displacements of mass m_1 and m_2 at any instance measured from equilibrium position respectively

Assuming $x_2 > x_1$

The differential equation of motion for the system may be expressed as:

$$\left.\begin{aligned} m_1\ddot{x}_1 &= -k_1 x_1 + k_2(x_2 - x_1) \\ m_2\ddot{x}_2 &= -k_2(x_2 - x_1) - k_3 x_2 \end{aligned}\right\} \text{---(1)}$$

or

$$\left.\begin{aligned} m_1\ddot{x}_1 + k_1 x_1 - k_2(x_2 - x_1) &= 0 \\ m_2\ddot{x}_2 + k_2(x_2 - x_1) + k_3 x_2 &= 0 \end{aligned}\right\} \text{---(2)}$$

or

$$\left.\begin{aligned} m_1\ddot{x}_1 + (k_1 + k_2)x_1 &= k_2 x_2 \\ m_2\ddot{x}_2 + (k_2 + k_3)x_2 &= k_2 x_1 \end{aligned}\right\} \text{---(3)}$$

Now assuming a solution for x_1 and x_2 under steady state conditions

$$\left.\begin{aligned} x_1 &= X_1 \sin\omega t \\ x_2 &= X_2 \sin\omega t \end{aligned}\right\} \text{---(4)}$$

Where X_1 and X_2 are the amplitude of two masses and ω is the frequency of harmonic motion.

From eq. (4)

$$x_1 = X_1 \sin\omega t \qquad x_2 = X_2 \sin\omega t$$
$$\dot{x}_1 = \omega X_1 \cos\omega t \qquad \dot{x}_2 = \omega X_2 \cos\omega t$$
$$\ddot{x}_1 = -\omega^2 X_1 \sin\omega t \qquad \ddot{x}_2 = -\omega^2 X_2 \sin\omega t \quad \text{— (5)}$$

substituting the values of eq. (5) in eq. (3) and cancelling common term $\sin\omega t$ at later stage

$$\left.\begin{aligned} -m_1\omega^2 X_1 \sin\omega t + (k_1+k_2) X_1 \sin\omega t &= k_2 X_2 \sin\omega t \\ -m_2\omega^2 X_2 \sin\omega t + (k_2+k_3) X_2 \sin\omega t &= k_2 X_1 \sin\omega t \end{aligned}\right\} \text{— (6)}$$

or
$$\left.\begin{aligned} \{-m_1\omega^2 + (k_1+k_2)\} X_1 &= k_2 X_2 \\ \{-m_2\omega^2 + (k_2+k_3)\} X_2 &= k_2 X_1 \end{aligned}\right\} \text{— (7)}$$

Eq. (7) gives two equations

$$\frac{X_1}{X_2} = \frac{k_2}{[(k_1+k_2) - m_1\omega^2]} \quad \text{— (8)}$$

$$\frac{X_1}{X_2} = \frac{(k_2+k_3) - m_2\omega^2}{k_2} \quad \text{— (9)}$$

Equating Eq. (8) and (9)

$$\frac{k_2}{[(k_1+k_2) - m_1\omega^2]} = \frac{[(k_2+k_3) - m_2\omega^2]}{k_2}$$

$$\Rightarrow k_2^2 = [(k_1+k_2) - m_1\omega^2][(k_2+k_3) - m_2\omega^2]$$

$$\Rightarrow m_1 m_2 \omega^4 - [m_1(k_2+k_3) + m_2(k_1+k_2)]\omega^2 + [k_1k_2 + k_1k_3 + k_2k_3] = 0 \quad \text{— (10)}$$

Eq. (10) gives two values of ω^2 and therefore two positive value of ω corresponding to the two natural frequencies ω_{n_1} and ω_{n_2} of the system. Eq. (10) is called frequency equation as the roots of this equation gives the natural frequencies of the system.

~~Now let~~ $\cancel{m_1 = m_2 = m}$ ~~and~~ $\cancel{k_1 = k_3 = k}$ ~~—(11)~~

~~Eq.(10) reduces to~~

$$\cancel{m^2\omega^4 - 2m(k+k_2)\omega^2 + (k^2+2kk_2)\,0}$$

~~which gives~~

$$\cancel{\omega_{n_1}, \omega_{n_2} = \sqrt{\frac{(k+k_2) \pm k_2}{m}}}$$

$$\Rightarrow \omega^4 - \left[\frac{k_1+k_3}{m_2} + \frac{k_1+k_2}{m_1}\right]\omega^2 + \frac{k_1k_2+k_1k_3+k_2k_3}{m_1m_2} = 0 \qquad \text{—(10)}$$

Equation (10) gives two values of ω^2 and therefore two possitive values of ω corresponding to the two natural frequencies ω_{n_1} and ω_{n_2} of the system. Eq.(10) is called the frequency equation as roots of this equation gives the natural frequency of the system.

Now let $m_1 = m_2 = m$ and $k_1 = k_3 = k$ $\Big\}$ —(11)

So equation (10) reduces to

$$\cancel{\omega^4 - \left[\frac{2K}{m} + \frac{k_1+k_2}{m}\right]\omega^2 + \frac{kk_2+k^2+k_2k}{m^2} = 0}$$

$$\cancel{\omega^4 - \left(\frac{2K}{m} + \frac{k_1+k_2}{m}\right)\omega^2 + \frac{2kk_2+k^2}{m^2} = 0}$$

$$\Rightarrow$$

$$\cancel{\omega^2 = 2K+}$$

$$m^2\omega^4 - 2m(k+k_2)\omega^2 + (k^2+2kk_2) = 0$$

which gives

$$\omega_{n_1}, \omega_{n_2} = \sqrt{\frac{(k+k_2) \pm k_2}{m}}$$

or

$$\left.\begin{aligned} \omega_{n_1} &= \sqrt{\frac{k}{m}} \\ \omega_{n_2} &= \sqrt{\frac{k+2k_2}{m}} \end{aligned}\right\} \text{—(12)}$$

substituting the condition of Eq.(11), in Eq.(8) and Eq.(9) can be reduced to

$$\frac{X_1}{X_2} = \frac{K_2}{\left[(K_1+K_2) - m\omega^2\right]} \quad \text{—} (13)$$

$$\frac{X_1}{X_2} = \frac{\left[(K_2+K) - m\omega^2\right]}{K_2} \quad \text{—} (14)$$

Now substituting the value of ω_{n_1} in eq.(12) in any of the Eq.(13) and Eq.(14), we have

$$\boxed{\frac{X_1}{X_2} = +1}$$

It means the system is vibrating with the first natural frequency ω_{n_1}, the mode shape is such that the ratio of amplitude is +1

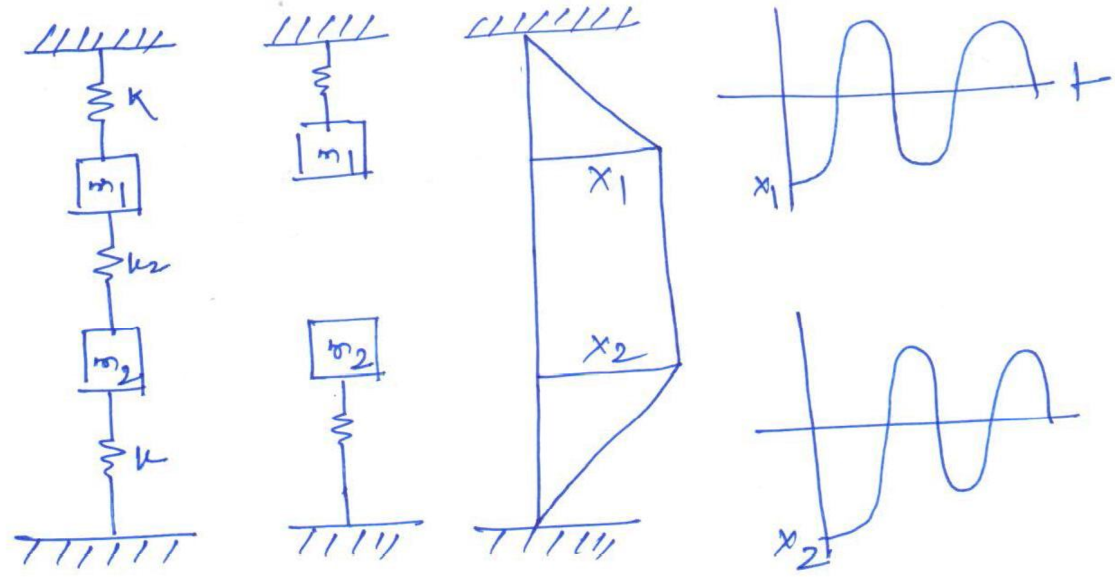
So $\left(\frac{X_1}{X_2}\right)_1 \rightarrow$ Ratio of amplitude in the first modeshape corresponding to first natural frequency ω_{n_1}

Now substituting the value of ω_{n_2} from eq.(12) in eq.(13) or eq.(14), we have

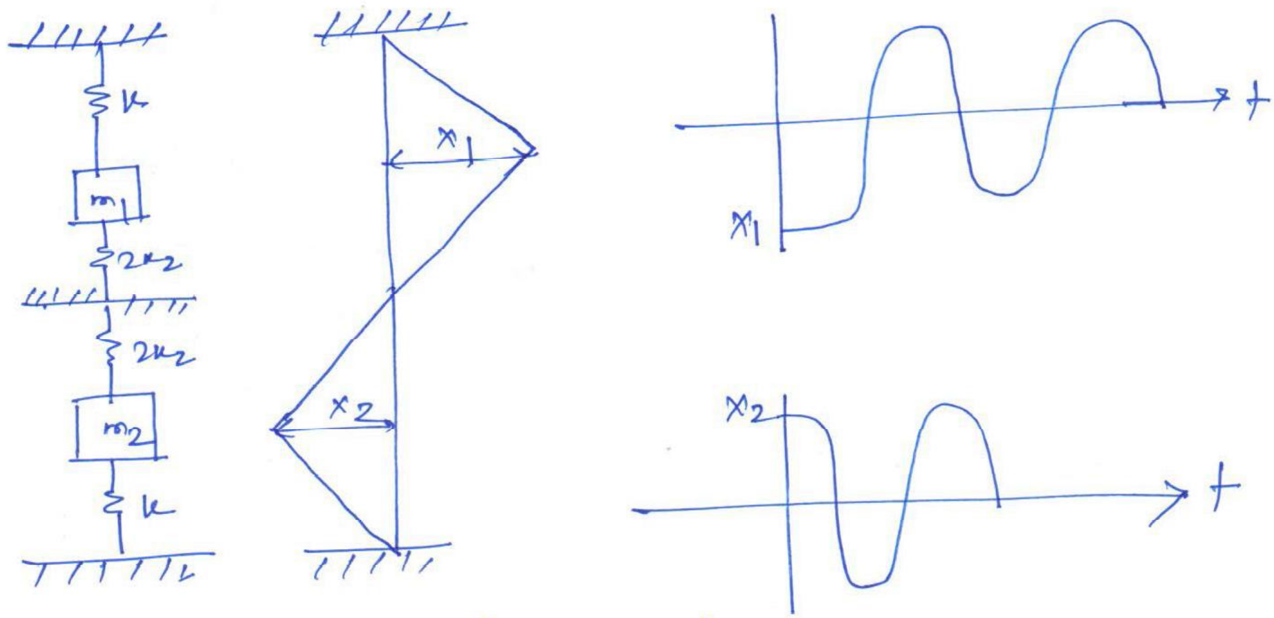
$$\boxed{\left(\frac{X_1}{X_2}\right) = -1}$$

and $\left(\frac{X_1}{X_2}\right)_2 \rightarrow$ indicates second mode shape corresponding to second natural frequency ω_{n_2}

- The ratio of amplitudes of two mass being +1, indicates the amplitudes are equal and two motions are in phase i.e the two masses move up and down together.
- The ratio of amplitude of two masses being −1 means the amplitudes are equal but the motions are out of phase i.e when the mass moving down the other mass is moving up and vice versa.



(1st mode)



(2nd Mode)

It can be seen that if the two masses are given equal initial displacement in the same direction and released they will vibrate in 1st principal mode of vibration with first natural frequency. Also if they are given equal initial displacement in opposite direction, and released they will vibrate in second principal mode of vibration with second natural frequency.

- However, if the the two masses are given unequal initial displacement in any direction their motion will be the super position of two harmonic motions corresponding to the two natural frequencies as:

$$\left.\begin{aligned} x_1 &= X_1' \cos \omega_{n_1} t + X_1'' \cos \omega_{n_1} t \\ x_2 &= X_2' \cos \omega_{n_2} t + X_2'' \cos \omega_{n_2} t \end{aligned}\right\} \quad -(15)$$

where X_1' and X_1'' $\rightarrow$ amplitudes of mass m_1 at lower and higher frequencies respectively

X_2' and X_2'' $\rightarrow$ amplitude of mass m_2 at lower and higher natural frequencies.

and they will have the relationship

$$\left.\begin{aligned} \left(\frac{X_1'}{X_2'}\right) &= \left(\frac{X_1}{X_2}\right)_1, \\ \left(\frac{X_1''}{X_2''}\right) &= \left(\frac{X_1}{X_2}\right)_2 \\ X_1' + X_1'' &= \text{initial displacement of } m_1 \\ X_2' + X_2'' &= \text{initial displacement of } m_2 \end{aligned}\right\} \quad -(16)$$

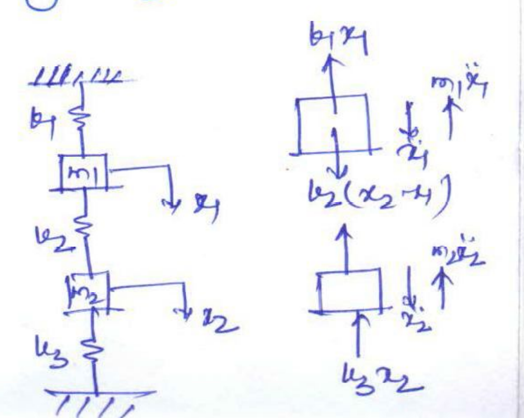
Example

For the system shown in the figure find two natural frequencies when:

$m_1 = m_2 = m = 9.8$ kg.

$k_1 = k_3 = 8820$ N/m

$k_2 = 3430$ N/m.



find out the resultant motions of m_1 and m_2 for the following different cases:

(a) both masses are displaced 5mm in downward directi and released simultaneously

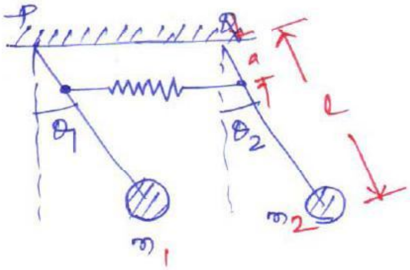
(b) both masses are displaced 5mm; m_1 in downward direction and m_2 in upward direction and released simultaneously

(c) mass m_1 is displaced 5mm downward and mass m_2 is displaced 7.5mm downward and released simultaneously

(d) mass m_1 displaced 5mm upward while m_2 is fixed and both ~~masses are~~ released simultaneously.

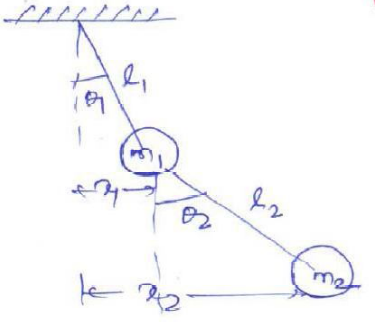
Assignment

1. Determine the normal modes of vibrations of the coupled pendulum as shown in the figure.

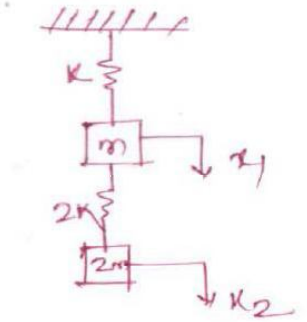


~~The equation of motion are.~~
Derive the equation of motion of the two masses and find natural frequencies of the system when
$k = 150$ N/m
$m_1 = 3$ kg , $m_2 = 5$ kg.
$l = 0.3$ m $a = 0.15$m.

Q.2 Setup the differential equations of motion for the double pendulum shown in the figure using coordinates x_1 and x_2 and assuming small amplitudes. Find the natural frequencies, ratios of amplitude and draw the mode shapes if $m_1 = m_2 = m$ and $l_1 = l_2 = l$

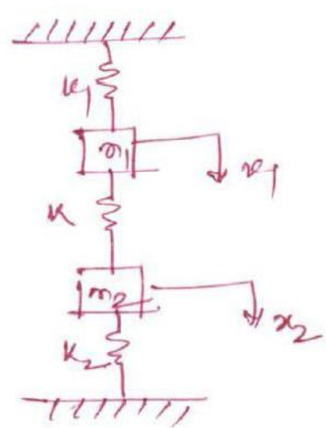


Q



Determine the natural frequency and amplitude of ratios of the system. Determine the response of the system at $K = 1000\,N/m$ and $m = 2\,Kg$.

Q.2



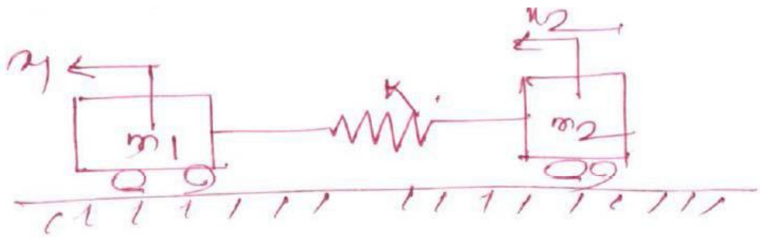
$k_1 = k_2 = 40\,N/m$

$K = 60\,N/m$,

$m_1 = m_2 = 10\,Kg$,

Determine the natural freq. of the system.

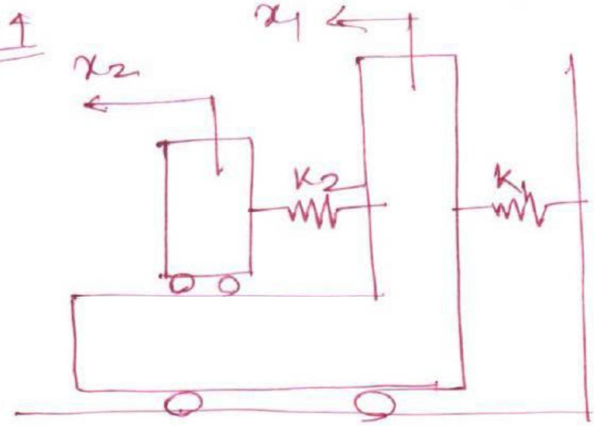
Q.3



$m_1 = 20\,kg$, $m_2 = 35\,kg$, $k = 300\,N/m$.

Determine natural frequency.

Q.4

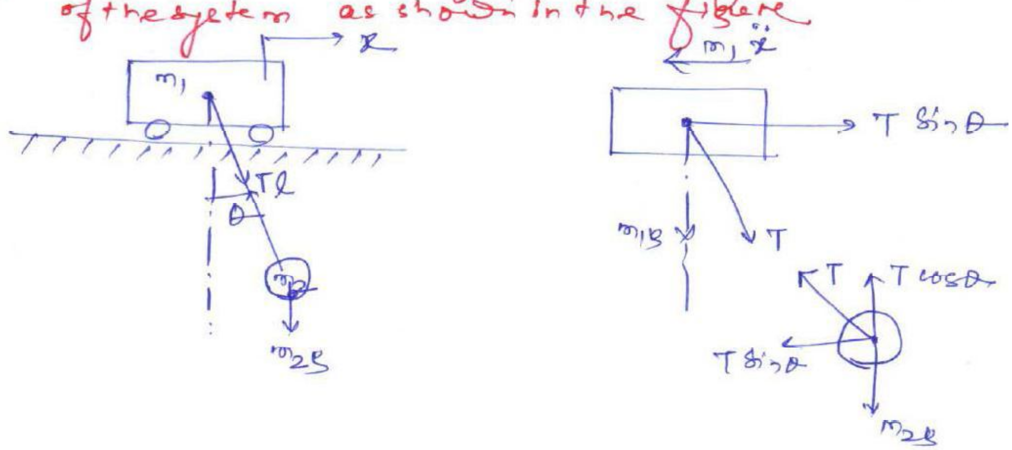


$m_1 = 200\,kg$ $m_2 = 50\,kg$.

$k_1 = 100,000\,N/m$

$k_2 = 200,000\,N/m$.

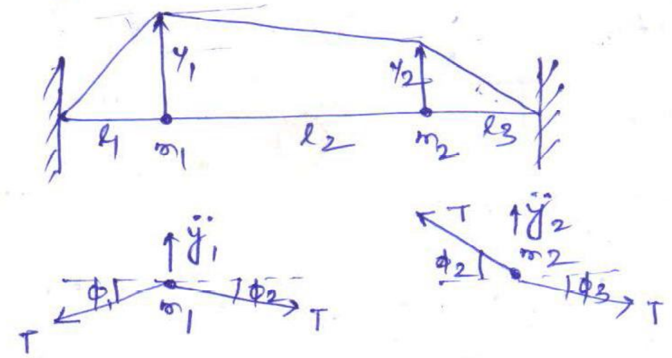
Q.3 Find the natural frequencies of vibration of the system as shown in the figure



Other cases of simple two dof systems:-

Different two dof systems are discussed in this section to find out the natural frequencies and corresponding mode shapes.

1. Two masses fixed on a tightly stretched string:-



consider two masses fixed on a tight string stretched between two supports and having tension T.

Let the amplitude of vibration is small and tension T is large.

At any instant let y_1 and y_2 be the displacement of two masses respectively.

The equation of lateral motion of the masses are:

$$\left.\begin{aligned} m_1\ddot{y}_1 + T\sin\phi_1 + T\sin\phi_2 = 0 \\ m_2\ddot{y}_2 - T\sin\phi_2 + T\sin\phi_3 = 0 \end{aligned}\right\} \quad (1)$$

Now we have

$$\left.\begin{aligned} \sin\phi_1 &= \frac{y_1}{l_1} \\ \sin\phi_2 &= \frac{y_1 - y_2}{l_2} \\ \sin\phi_3 &= \frac{y_2}{l_3} \end{aligned}\right\} \quad (2)$$

substituting the values of eq. (2) in eq. (1)

$$m_1\ddot{y}_1 + T\frac{y_1}{l_1} + T\left(\frac{y_1 - y_2}{l_2}\right) = 0$$

$$m_2\ddot{y}_2 - T\left(\frac{y_1 - y_2}{l_2}\right) + T\frac{y_2}{l_3} = 0$$

or

$$\left.\begin{aligned} m_1\ddot{y}_1 + \left(\frac{T}{l_1} + \frac{T}{l_2}\right)y_1 &= \frac{T}{l_2}y_2 \\ m_2\ddot{y}_2 + \left(\frac{T}{l_2} + \frac{T}{l_3}\right)y_2 &= \frac{T}{l_2}y_1 \end{aligned}\right\} \quad (3)$$

Assuming a steady state solution for principal mode vibration

$$\left.\begin{aligned} y_1 &= Y_1 \sin \omega t \\ y_2 &= Y_2 \sin \omega t \end{aligned}\right\} \quad \text{— (4)}$$

~~Subs~~ from eq.(4) we have

$$\dot{y}_1 = \omega Y_1 \cos\omega t \qquad \dot{y}_2 = \omega Y_2 \cos\omega t$$

$$\ddot{y}_1 = -\omega^2 Y_1 \sin\omega t \qquad \ddot{y}_2 = -\omega^2 Y_2 \sin\omega t$$

substituting the value in eq.(3)

$$-m_1\omega^2 Y_1 \sin\omega t + \left(\frac{T}{l_1}+\frac{T}{l_2}\right) Y_1 \sin\omega t = \frac{T}{l_2} Y_2 \sin\omega t$$

$$-m_2\omega^2 Y_2 \sin\omega t + \left(\frac{T}{l_2}+\frac{T}{l_3}\right) Y_2 \sin\omega t = \frac{T}{l_2} Y_1 \sin\omega t$$

or

$$\left.\begin{aligned} \left[-m_1\omega^2 + \left(\frac{T}{l_1}+\frac{T}{l_2}\right)\right] Y_1 &= \frac{T}{l_2} Y_2 \\ \left[-m_2\omega^2 + \left(\frac{T}{l_2}+\frac{T}{l_3}\right)\right] Y_2 &= \frac{T}{l_2} Y_1 \end{aligned}\right\} \quad \text{— (5)}$$

from eq.(5) the ratio of amplitude of vibration can be obtained as

$$\frac{Y_1}{Y_2} = \frac{T/l_2}{\left[\left(\frac{T}{l_1}+\frac{T}{l_2}\right) - m_1\omega^2\right]} \quad \text{— (6)}$$

$$\frac{Y_1}{Y_2} = \frac{\left[\left(\frac{T}{l_2}+\frac{T}{l_3}\right) - m_2\omega^2\right]}{T/l_2} \quad \text{— (7)}$$

frequency equation can be obtained by equating eq.(6) and (7)

$$\frac{T/l_2}{\left(\frac{T}{l_1}+\frac{T}{l_2}\right) - m_1\omega^2} = \frac{\left(\frac{T}{l_2}+\frac{T}{l_3}\right) - m_2\omega^2}{T/l_2}$$

$$\left[\left(\frac{T}{l_1}+\frac{T}{l_2}\right)-m_1\omega^2\right]\left[\left(\frac{T}{l_2}+\frac{T}{l_3}\right)-m_2\omega^2\right]=\frac{T^2}{l_2^2}$$

$$\Rightarrow \quad m_1m_2\omega^4-\left[m_1\left(\frac{T}{l_2}+\frac{T}{l_3}\right)+m_2\left(\frac{T}{l_1}+\frac{T}{l_2}\right)\right]\omega^2 + \frac{T^2}{l_1l_2}+\frac{T^2}{l_1l_3}+\frac{T^2}{l_2l_3}=0 \qquad \text{—— (8)}$$

Assuming

$$\left.\begin{aligned} m_1 = m_2 &= m \\ l_1=l_2=l_3&=l \end{aligned}\right\} \text{—— (9)}$$

We have

$$m^2\omega^4-\left[m\left(\frac{2T}{l}\right)+m\left(\frac{2T}{l}\right)\right]\omega^2+\frac{3T^2}{l^2}=0$$

or

$$m^2\omega^4-\frac{4mT}{l}\omega^2+\frac{3T^2}{l^2}=0 \qquad \text{—— (10)}$$

solving for ω, the two values of natural frequencies are

$$\left.\begin{aligned} \omega_{n_1} &= \sqrt{\frac{T}{ml}} \\ \omega_{n_2} &= \sqrt{\frac{3T}{ml}} \end{aligned}\right\} \text{—— (11)}$$

The ratio of vibration amplitude can be expressed as.

$$\left.\begin{aligned} \frac{Y_1}{Y_2} &= \frac{T/l}{\frac{2T}{l}-m\omega^2} \\ \frac{Y_1}{Y_2} &= \frac{\frac{2T}{l}-m\omega^2}{T/l} \end{aligned}\right\} \text{—— (12)}$$

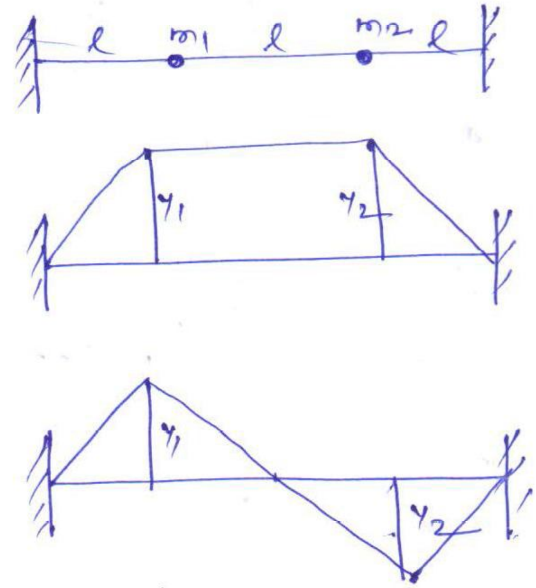
or

$$\frac{T/l}{\frac{2T}{l}-m\omega^2}=\frac{2T/l-m\omega^2}{T/l}$$

$$\Rightarrow \quad \frac{2T}{l}-m\omega^2=\mp\frac{T}{l}$$

$$\left.\begin{aligned} \omega_{n_1} &= \sqrt{\frac{T}{ml}} \\ \omega_{n_2} &= \sqrt{\frac{3T}{ml}} \end{aligned}\right\} \text{—— (11)}$$

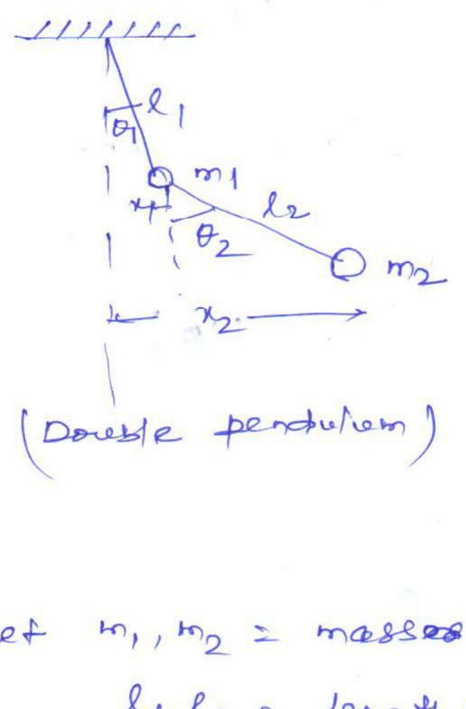
The corresponding principal mode shapes are obtained by substituting in either of the equation (10) the values ω_{n1} and ω_{n2}



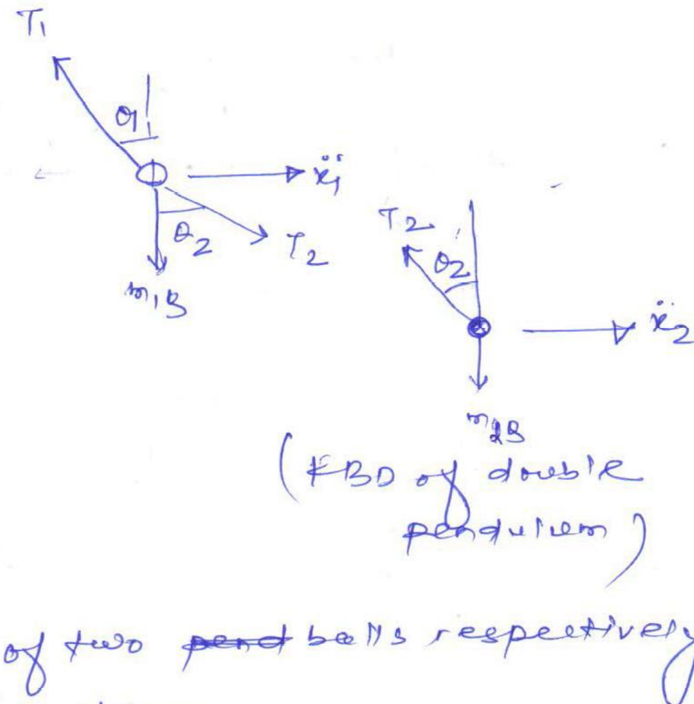
$$\left(\frac{y_1}{y_2}\right)_1 = +1$$

$$\left(\frac{y_1}{y_2}\right)_2 = -1$$

Double Pendulum :-



(Double pendulum)



(FBD of double pendulum)

Let m_1, m_2 = masses of two ~~pend~~ balls respectively

l_1, l_2 = length of strings.

From the geometry
$$\left.\begin{aligned}\sin\theta_1 &= \frac{x_1}{l_1}\\ \sin\theta_2 &= \frac{x_2 - x_1}{l_2}\end{aligned}\right\} \quad \text{--- (1)}$$

Considering no vertical motion and resolving the vertical components,

$$\left.\begin{aligned} T_2\cos\theta_2 &= m_2 g \\ T_1\cos\theta_1 &= m_1 g + T_2\cos\theta_2 \end{aligned}\right\} \quad -(2)$$

For small values of θ_1 and θ_2

$$\left.\begin{aligned} T_2 &= m_2 g \\ T_1 &= m_1 g + T_2 = (m_1+m_2) g \end{aligned}\right\} \quad -(3)$$

Now the differential equation of motion of the two masses in horizontal direction

$$\left.\begin{aligned} m_1\ddot{x}_1 + T_1\sin\theta_1 - T_2\sin\theta_2 &= 0 \\ m_2\ddot{x}_2 + T_2\sin\theta_2 &= 0 \end{aligned}\right\} \quad -(4)$$

substituting the values of T_1 and T_2 and $\sin\theta_1$ and θ_2 in above equation we have

$$m_1\ddot{x}_1 + (m_1+m_2) g \cdot \frac{x_1}{l_1} - m_2 g\left(\frac{x_2-x_1}{l_2}\right) = 0$$

$$m_2\ddot{x}_2 + m_2 g\left(\frac{x_2-x_1}{l_2}\right) = 0$$

or

$$\left.\begin{aligned} m_1\ddot{x}_1 + \left[\frac{(m_1+m_2)}{l_1} + \frac{m_2}{l_2}\right] x_1 g &= \frac{m_2}{l_2} \cdot x_2 g \\ m_2\ddot{x}_2 + \frac{m_2}{l_2} g x_2 &= \frac{m_2}{l_2} g x_1 \end{aligned}\right\} \quad -(5)$$

Assuming a steady solution for the principal mode of vibration

$$\left.\begin{aligned} x_1 &= X_1 \sin\omega t \\ x_2 &= X_2 \sin\omega t \end{aligned}\right\} \quad -(6)$$

From equation (6)

$$\dot{x}_1 = \omega X_1 \cos\omega t \qquad \dot{x}_2 = \omega X_2 \cos\omega t$$

$$\ddot{x}_1 = -\omega^2 X_1 \sin\omega t \qquad \ddot{x}_2 = -\omega^2 X_2 \sin\omega t \quad -(7)$$

substituting the values of $x_1, \dot{x}_1, \ddot{x}_1$ and $x_2, \dot{x}_2, \ddot{x}_2$ in equation (5), we have

$$-m_1\omega^2 X_1 \sin\omega t + \left[\frac{m_1+m_2}{l_1} + \frac{m_2}{l_2}\right] g \cdot X_1 \sin\omega t = \frac{m_2}{l_2} g \cdot X_2$$

$$-m_2\omega^2 X_2 \sin\omega t + \frac{m_2}{l_2} g\, X_2 \sin\omega t = \frac{m_2}{l_2} g\, X_1 \sin\omega t \qquad \text{--- (8)}$$

cancelling out the common term of $\sin\omega t$ from the equation

$$\left\{\begin{aligned} &\left[-m_1\omega^2 + \left[\frac{m_1+m_2}{l_1} + \frac{m_2}{l_2}\right] g\right] X_1 = \frac{m_2}{l_2} g X_2 \\ &\left[-m_2\omega^2 + \frac{m_2}{l_2} g\right] X_2 = \frac{m_2}{l_2} g\, X_1 \end{aligned}\right\} \text{--- (9)}$$

From equation (9) we have two values of $\frac{X_1}{X_2}$ as:

$$\boxed{\frac{X_1}{X_2} = \frac{m_2/l_2 \cdot g}{\left[\frac{m_1+m_2}{l_1} + \frac{m_2}{l_2}\right] g - m_1\omega^2}} \text{ --- (10)}$$

$$\boxed{\frac{X_1}{X_2} = \frac{\left(\frac{m_2}{l_2} g - m_2\omega^2\right)}{\frac{m_2}{l_2} g}} \text{ --- (11)}$$

considering a special case of $m_1 = m_2 = m$
and $l_1 = l_2 = l$

Equation (10) and (11) may be written as

$$\frac{X_1}{X_2} = \frac{m/l \cdot g}{\left(\frac{2m}{l} + \frac{m}{l}\right) g - m\omega^2} = \frac{m/l \cdot g}{\frac{3m}{l} g - m\omega^2}$$

$$\Rightarrow \boxed{\frac{X_1}{X_2} = \frac{g/l}{\left(\frac{3g}{l} - \omega^2\right)}} \text{ --- (12)}$$

And $$\boxed{\frac{X_1}{X_2} = \frac{\left(\frac{m}{l} g - m\omega^2\right)}{\frac{mg}{l}} = \frac{\left(g/l - \omega^2\right)}{\left(g/l\right)}} \text{ --- (13)}$$

Equating equation (12) and (13)

$$\frac{g/l}{\frac{3g}{l} - \omega^2} = \frac{g/l - \omega^2}{g/l}$$

$$\Rightarrow \frac{3g^2}{l^2} - \frac{3g\omega^2}{l} - \frac{g\omega^2}{l} + \omega^4 = \frac{g^2}{l^2}$$

$$\Rightarrow \boxed{\omega^4 - \frac{4g\omega^2}{l} + \frac{2g^2}{l^2} = 0} \quad \text{---(14)}$$

or

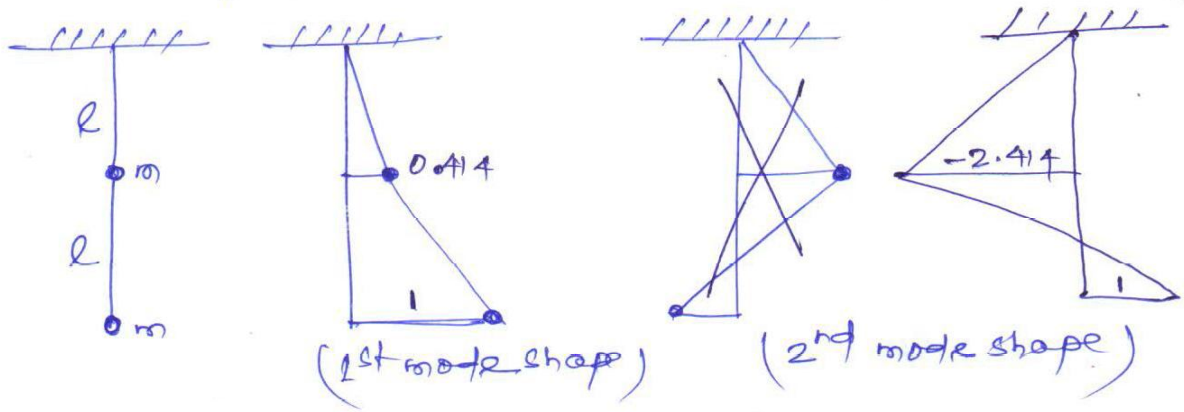
$$\left.\begin{aligned} \omega_{n_1} &= \sqrt{\frac{g}{l}\left(2-\sqrt{2}\right)} \\ \omega_{n_2} &= \sqrt{\frac{g}{l}\left(2+\sqrt{2}\right)} \end{aligned}\right\} \quad \text{---(15)}$$

corresponding modeshapes can be obtained by substituting the value of ω_{n_1} and ω_{n_2} in equations (12) and (13) for 1st and 2nd mode shapes respectively

So the principal modes are:

$$\boxed{\begin{aligned} \left(\frac{x_1}{x_2}\right)_1 &= \frac{1}{1+\sqrt{2}} = -1+\sqrt{2} = +0.414 \\ \left(\frac{x_1}{x_2}\right)_2 &= \frac{1}{1-\sqrt{2}} = -1-\sqrt{2} = -2.414 \end{aligned}} \quad \text{---(16)}$$

The mode shapes are as shown in the figure:



(1st mode shape) (2nd mode shape)